Homework #6

Due next Wed

1. Find the solution for the indicated initial conditions:

$$\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix}, \; y(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

2. Find the solution for the indicated initial conditions:

$$\begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix}, \; y(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

3. Find the solution for the indicated initial conditions:

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -4 \\ -3 & 9 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}, \; y(0) = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

4. Prove that A*B = B*A for any square, diagonal matrices A, B.

5. Show that, for any square matrix A: (i.e. A commutes with e^{tA})

$$\frac{de^{tA}}{dt} = Ae^{tA} = e^{tA}A$$

Homework #6 Solutions

1. Find the solution for the given initial conditions

$$\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \quad y(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

find the eigenvalues

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$= -(2-\lambda)(2+\lambda) - 1 = -(4-\lambda^2) - 1 = -5 + \lambda^2 = 0$$

$$\lambda^2 = 5$$

$$\lambda = \pm\sqrt{5}$$

Find the eigenvectors

$\lambda = \sqrt{5}$

$$\begin{bmatrix} -2-\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -\sqrt{5}+2 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ the ~~eigenvalue~~ eigenvector is $\begin{pmatrix} -\sqrt{5}+2 \\ -1 \end{pmatrix}$

$\lambda = -\sqrt{5}$

$$\begin{bmatrix} -2+\sqrt{5} & 1 \\ 1 & 2+\sqrt{5} \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & \sqrt{5}+2 \\ 0 & 0 \end{bmatrix}$$

~~the eigenvector is~~

$$\begin{bmatrix} 1 & \sqrt{5}+2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{5}+2 \\ 0 & -1 \end{bmatrix}$$

$\therefore$ the eigenvector is $\begin{pmatrix} \sqrt{5}+2 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 \begin{pmatrix} -\sqrt{5}+2 \\ -1 \end{pmatrix} e^{\sqrt{5}t} + C_2 \begin{pmatrix} \sqrt{5}+2 \\ -1 \end{pmatrix} e^{-\sqrt{5}t}$$

$$y(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$y(0) = \begin{bmatrix} y(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} -\sqrt{5}+2 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} \sqrt{5}+2 \\ -1 \end{bmatrix}$$

$$1 = -C_1 - C_2 \qquad C_1 = -1 - C_2$$

$$4 = C_1(-\sqrt{5}+2) + C_2(\sqrt{5}+2) = -(1+C_2)(-\sqrt{5}+2) + C_2(\sqrt{5}+2)$$

#1 (cont)

$$4 = C_2(\sqrt{5}+2+\sqrt{5}-2) - 2 + \sqrt{5}$$

$$C_2 = \frac{4+2-\sqrt{5}}{2\sqrt{5}} = .84$$

$$C_1 = -1 - C_2 = -1.84$$

$$\boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -1.84\begin{bmatrix} -\sqrt{5}+2 \\ -1 \end{bmatrix} e^{\sqrt{5}t} + .84\begin{bmatrix} \sqrt{5}+2 \\ -1 \end{bmatrix} e^{-\sqrt{5}t}}$$

#2

$$\begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \quad y(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

find eigenvalues

$$\begin{vmatrix} -5-\lambda & 1 \\ -6 & -\lambda \end{vmatrix} = 0 = 5\lambda + \lambda^2 + 6 = (\lambda+3)(\lambda+2)$$

$$\lambda = -3, -2$$

find eigenvectors

$\lambda = -3$

$$\begin{bmatrix} -5+3 & 1 \\ -6 & +3 \end{bmatrix} \to \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

∴ the eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -2$

$$\begin{bmatrix} -5+2 & 1 \\ -6 & +2 \end{bmatrix} \to \begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

∴ the eigenvector is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-3t} + C_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-2t}$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$C_1 + C_2 = 3$$

$$2C_1 + 3C_2 = 2 = 2C_1 + 3(3 - C_1)$$

#2 (cont)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-3t} + -4\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-2t}$$

#3

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -4 \\ -3 & 9 & -7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} \; ; \quad y(0) = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

find eigenvalues

$$\begin{vmatrix} -\lambda & -1 & 0 \\ -1 & 4-\lambda & -4 \\ -3 & 9 & -7-\lambda \end{vmatrix} = 0 = -(\lambda^3 + 3\lambda^2 + 7\lambda + 5)$$

$$0 = -(\lambda+1)(\lambda^2 + 2\lambda + 5)$$

$$\lambda = -1, -1-2i, -1+2i$$

find eigenvectors

$\lambda = -1$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 5 & -4 \\ -3 & 9 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = -1-2i$

$$\begin{bmatrix} 1+2i & -1 & 0 \\ -1 & 5+2i & -4 \\ -3 & 9 & -6+2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{3}i \\ 0 & 1 & -\frac{2}{3}+\frac{1}{3}i \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ the eigenvector is $\begin{bmatrix} \frac{1}{3}i \\ -\frac{2}{3}+\frac{1}{3}i \\ -1 \end{bmatrix}$

$\lambda = -1+2i$

$$\begin{bmatrix} 1-2i & -1 & 0 \\ -1 & 5-2i & -4 \\ -3 & 9 & -6-2i \end{bmatrix} \rightarrow \begin{bmatrix} -\frac{1}{3}i \\ -\frac{2}{3}-\frac{1}{3}i \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ \end{bmatrix} \quad \begin{bmatrix} 1 \\ \end{bmatrix} e^{-t} \quad \begin{bmatrix} \frac{1}{3}i \\ \end{bmatrix} e^{-(1+2i)t} \quad \begin{bmatrix} -\frac{1}{3}i \\ \end{bmatrix} e^{(-1+2i)t}$$

#3 (cont)

$$\underline{y}(0) = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{3}i \\ -\frac{2}{3} + \frac{1}{3}i \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -\frac{1}{3}i \\ -\frac{2}{3} - \frac{1}{3}i \\ -1 \end{bmatrix}$$

Solving for c_1, c_2, c_3 gives

$$c_1 = -1$$
$$c_2 = -\frac{3}{2} + \frac{3}{2}i$$
$$c_3 = -\frac{3}{2} - \frac{3}{2}i$$

$$\boxed{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-t} + \left(-\frac{3}{2} + \frac{3}{2}i\right) \begin{bmatrix} \frac{1}{3}i \\ -\frac{2}{3} + \frac{1}{3}i \\ -1 \end{bmatrix} e^{-(1+2it)} + \left(\frac{3}{2} + \frac{3}{2}i\right) \begin{bmatrix} \frac{1}{3}i \\ \frac{2}{3} + \frac{1}{3}i \\ 1 \end{bmatrix} e^{(-1+2i)t}}$$

Prove $A*B = B*A$ for any square, diagonal matrices A, B

$$\begin{bmatrix} a_1 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & \cdots \\ 0 & 0 & a_3 & \\ \vdots & \vdots & & \ddots \end{bmatrix}\begin{bmatrix} b_1 & 0 & 0 & \cdots \\ 0 & b_2 & 0 & \cdots \\ 0 & 0 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

try for a 2×2 matrix

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{bmatrix}$$

$$\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{bmatrix}$$

generalize

$A*B = C$

$C_{ij} = \sum_k A_{ik} B_{kj}$

$A_{ij} = 0$ for $i \neq j$
$B_{ij} = 0$ for $i \neq j$

$\therefore \sum_k A_{ik} B_{kj} = 0$ for $k \neq i$ or $k \neq j$

$\sum_k A_{ik} B_{kj} \neq 0$ for $k = i = j$

$\therefore C_{ij} = 0$ for $i \neq j$

$C_{ij} = A_{ii} B_{jj}$ for $i = j$

$$\therefore A*B = \begin{bmatrix} a_1b_1 & 0 & \cdots \\ 0 & a_2b_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$B*A = \begin{bmatrix} a_1b_1 & 0 & \cdots \\ 0 & a_2b_2 & \cdots \\ & & \ddots \end{bmatrix}$$

$\boxed{\therefore A*B = B*A}$

5. Show that for any square matrix A

$$\frac{d e^{tA}}{dt} = Ae^{tA} = e^{tA} A$$

A is a square matrix

$$e^{At} = I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 \ldots$$

$$\frac{d e^{At}}{dt} = A + tA^2 + \frac{1}{2!}t^2A^3 + \frac{1}{3!}t^3A^4 \ldots = Ae^{At} = e^{At}A$$

because A will commute w/ itself