CHAPTER 2

FUNCTIONS OF A COMPLEX VARIABLE

2.1 COMPLEX NUMBERS

In the course of study of roots of algebraic equations and in particular the cubic equation, it has been found convenient to introduce the concept of a number whose square is equal to -1. By a well-established tradition, this number is denoted by i, and we write $i^2 = -1$ and $i = \sqrt{-1}$. If we allow i to be multiplied by real numbers, we obtain the so-called *imaginary numbers** of the form bi (where b is real). If the usual rules of multiplication are extended to imaginary numbers, then we must conclude that the products of imaginary numbers are real numbers; moreover, their squares are negative real numbers. For instance,

$$(3i)(-4i) = (3)(-4)i^2 = (-12)(-1) = 12,$$
$$(-5i)^2 = (-5)^2 i^2 = -25.$$

If imaginary numbers are adjoined to real numbers, we have a system within which we can perform multiplication and division (except by zero, of course). We say that such a system is *closed* under multiplication and division. However, our system is not closed under addition and subtraction.† To eliminate this deficiency, so-called *complex numbers* are introduced. These are numbers which are most often written in the form

$$a + bi \qquad (a, b = \text{real numbers})$$

and are assumed to obey appropriate algebraic rules. As will be shown below, the system of complex numbers is closed under addition, subtraction, multiplication, and division plus the "extraction of roots" operation. In short, it has all the desirable algebraic characteristics and represents an extension of the real number system. The study of complex numbers is invaluable for every physicist because the description of physical laws is much more complicated without them.

* Imaginary numbers are also called *pure imaginary numbers* to stress the distinction from the more general case of complex numbers. The name originated from the belief that imaginary numbers, as well as complex numbers, do not represent directly observable quantities in nature. While this point of view is now mostly abandoned, the original nomenclature still exists.

† The system is not closed under the operation of extraction of the square root either; for example, $\sqrt{i}$ is neither real nor (pure) imaginary.

2.2 BASIC ALGEBRA AND GEOMETRY OF COMPLEX NUMBERS

If complex numbers are written in the usual form $a + ib$ (or $a + bi$) then the usual algebraic operations with them are defined as follows.

1. Addition:

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2).$$

2. Multiplication:

$$(a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).$$

The second rule is easy to follow if we recognize that the expressions $a + ib$ are multiplied in the same manner as binomials, using the distributive and associative laws, and i^2 is replaced by -1.

Complex numbers of the form $a + i0$ are tacitly identified with real numbers since they obey the same algebraic rules and are generally indistinguishable from each other.* Complex numbers of the form $0 + ib$ are then (pure) imaginary numbers. It is customary to write simply $a + i0 = a$ and $0 + ib = ib$. Subtraction of complex numbers can be defined as *inverse addition* so that if

$$(a_1 + ib_1) - (a_2 + ib_2) = x + iy,$$

then

$$a_1 + ib_1 = (x + iy) + (a_2 + ib_2)$$

from which it follows that†

$$x = a_1 - a_2 \quad \text{and} \quad y = b_1 - b_2.$$

An alternative is to form the negative of a complex number,

$$-(a + ib) = (-1)(a + ib) = (-1 + i0)(a + ib) = -a - ib,$$

and reduce the subtraction to addition.

The rule for division can be similarly deduced by inverting the multiplication. A shortcut method is given by the following technique:

$$\frac{a + ib}{c + id} = \frac{a + ib}{c + id}\,\frac{c - id}{c - id} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} + i\,\frac{bc - ad}{c^2 + d^2} \qquad (c^2 + d^2 \neq 0).$$

It is readily seen that the divisor can be any complex number except zero (namely the number $0 + i0$, which is unique and is written simply 0).

* In a more rigorous language, "the subset of complex numbers of the form $a + i0$ is *isomorphic* to the set of real numbers under the correspondence $a + i0 \leftrightarrow a$."

† It is tacitly postulated that $x_1 + iy_1 = x_2 + iy_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Remarks

1. The addition of complex numbers obeys the same rule as the addition of vectors in plane, provided a and b are identified with components of a vector. Note, however, that the multiplication of complex numbers differs from the formation of dot and cross products of vectors.

2. The use of the symbol i and the related binomial $a + ib$ is conventional, but not indispensable. It is possible to define a complex number as a pair of real numbers, (a, b), obeying certain peculiar rules, e.g., the multiplication can be defined by

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1),$$

and so on. It should be clear that the form $a + ib$ is just a *representation* of a complex number.

It is customary to represent complex numbers by points in the so-called *complex plane*, or Argand diagram (Fig. 2.1). If we denote the complex number $x + iy$ by a single symbol z and write $z = x + iy$, then to each z there corresponds a point in the complex plane with the abscissa x and the ordinate y. This idea also leads us to the *trigonometric representation* of a complex number:

$$z = r(\cos\theta + i\sin\theta),$$

where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$. In this representation r is unique (positive square root) but θ is not. A common convention is to demand that†

$$-\pi < \theta \leq \pi,$$

along with the standard rule of quadrants, namely, $\theta < 0$ if $y < 0$.

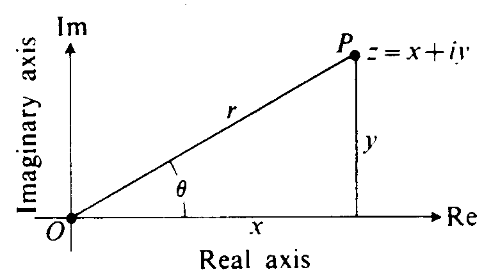

Figure 2.1

The following nomenclature and notation will be widely used: If

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

then

$x = \text{Re}\, z$ is the *real part* of z,

$y = \text{Im}\, z$ is the *imaginary part* of z,

$r = |z|$ is the *modulus* of z, also known as the *magnitude* or *absolute value* of z,

θ is the *argument* of z, also called the *polar angle* or *phase*.‡

The number $x - iy$ is called the *complex conjugate* of the number $z = x + iy$ and vice versa. We shall denote it by z^*. We can say that z and z^* represent (on the complex plane) the reflections of each other with respect to the real axis.

† Another commonly used convention is $0 \leq \theta < 2\pi$.

‡ A more precise name for θ would be the "principal value of the argument of z" (see p. 57).

Remarks

1. The quantity zz^* is always a nonnegative real number equal to $|z|^2$ or to $|z^*|^2$ (which are the same).

2. The quantity $z + z^*$ is always a real number, equal to $2 \operatorname{Re} z$ or to $2 \operatorname{Re} z^*$ (which are the same).

3. The rules $(z_1 + z_2)^* = z_1^* + z_2^*$ and $(z_1 z_2)^* = z_1^* z_2^*$ are evident and should be remembered.

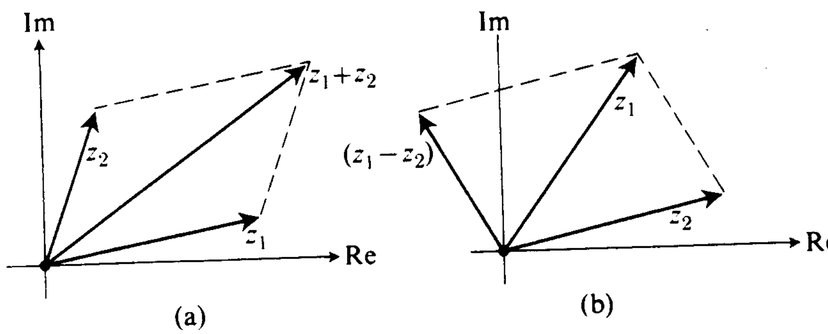

Figure 2.2

Because complex numbers obey the same addition rule that applies to vectors in a plane, they can be added graphically by the *parallelogram rule* (Fig. 2.2a). Conversely, vectors in a plane can be represented by complex numbers. The scalar product of two such vectors can be obtained by the rule

$$(\mathbf{z}_1 \cdot \mathbf{z}_2) = \operatorname{Re} (z_1^* z_2) = \operatorname{Re} (z_1 z_2^*),$$

where it is understood that $\mathbf{z}_1$ and $\mathbf{z}_2$ are vectors corresponding to complex numbers z_1 and z_2 respectively. The vector product can be obtained in a similar fashion:

$$[\mathbf{z}_1 \times \mathbf{z}_2] = \operatorname{Im} (z_1^* z_2) = -\operatorname{Im} (z_1 z_2^*).$$

Exercise. Verify the validity of the above rules for scalar and vector products.

In the theory of complex variables, the expression $|z_1 - z_2|$ is often used. According to Fig. 2.2(b) this quantity (modulus of the complex number $z_1 - z_2$) is equal to the distance between the points z_1 and z_2 in the complex plane. It follows that the statement $|z - z_0| < R$ (which often occurs in proofs of various theorems) means geometrically that point z is within the circle of radius R drawn around the point z_0 as a center (i.e., z is in the R-neighborhood of z_0; see p. 16). The following two inequalities are easily proved from geometrical considerations:

1. $|z_1 + z_2| \leq |z_1| + |z_2|.$

(A side of a triangle is less than or equal to the sum of the other two sides.)

2. $|z_1 - z_2| \geq ||z_1| - |z_2||.$

(The difference of two sides of a triangle is less than or equal to the third side.)

Remark. It should be emphasized that inequalities can exist only among the *moduli* of complex numbers, not among the complex numbers themselves. A complex number cannot be *greater* or *smaller* than another complex number. Also, there are no *positive* or *negative* complex numbers.

2.3 DE MOIVRE FORMULA AND THE CALCULATION OF ROOTS

While addition and subtraction of complex numbers are most easily performed in their cartesian form $z = x + iy$, multiplication and division are easier in trigonometric form. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then elementary calculation shows that

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

with the provision that if $\theta_1 + \theta_2$ happens to be greater than π, or less than or equal to $-\pi$, then the amount 2π should be added or subtracted to fulfill the condition $-\pi < (\theta_1 + \theta_2) \leq \pi$.

Remark. It should be emphasized that even though $\cos(\theta \pm 2\pi) = \cos\theta$ and $\sin(\theta \pm 2\pi) = \sin\theta$, the value of θ is supposed to be uniquely specified. This will become evident when θ is subjected to certain operations, e.g., in the course of evaluation of roots. The convention $-\pi < \theta \leq \pi$ is not the only one possible, but *some* convention must be adopted and ours is just as good as any other.

Using the same trigonometric identities as in the above multiplication rule, we can also obtain the so-called *De Moivre formula:*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad (n = \text{integer}).$$

Thus we now have the general rule for calculating the nth power of a complex number z. If $z = r(\cos\theta + i\sin\theta)$, then $z^n = R(\cos\phi + i\sin\phi)$, where $R = r^n$ and $\phi = n\theta \pm 2\pi k$ with the integer k chosen in such a way that $-\pi < \phi \leq \pi$.

The rule for calculating the nth root of a complex number can now be derived without much difficulty. If $z = r(\cos\theta + i\sin\theta)$, then the complex number

$$w_0 = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

is definitely the nth root of z because $w_0^n = z$. However, this is not the only nth root of z; the numbers

$$w_k = \sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right),$$

where $k = 1, 2, 3, \ldots, (n - 1)$, are also nth roots of z because $w_k^n = z$. It is

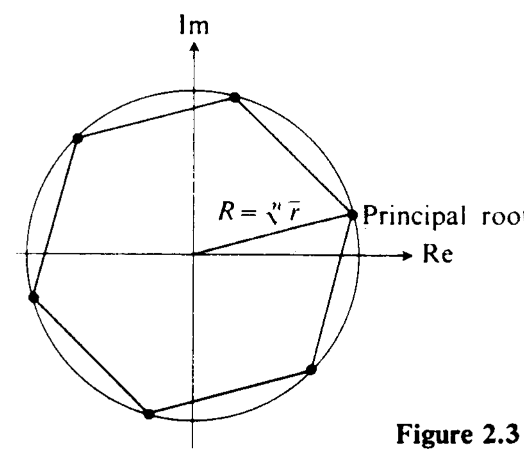

Figure 2.3

customary to call the number w_0 the *principal root* of z. The nth roots of a complex number z are always located at the vertices of a regular polygon of n sides inscribed in a circle of radius $R = \sqrt[n]{r}$ about the origin (Fig. 2.3).

Exercise. Verify that all possible roots of a complex number z are given by the above formulas. Show that all complex numbers except one have exactly n (different) nth-order roots. Which complex number is the exception?

2.4 COMPLEX FUNCTIONS. EULER'S FORMULA

Complex numbers $z = x + iy$ may be considered as variables if x or y (or both) vary. If this is so, then complex functions may be formed. For instance, consider the equation $w = z^2$. If we write $z = x + iy$ and $w = u + iv$, it follows that

$$u = x^2 - y^2, \qquad v = 2xy.$$

From this we conclude that if w is a function of z, then u and v are, in general, functions of both x and y. Thus we are dealing with two (independent) real functions of two (independent) real variables.

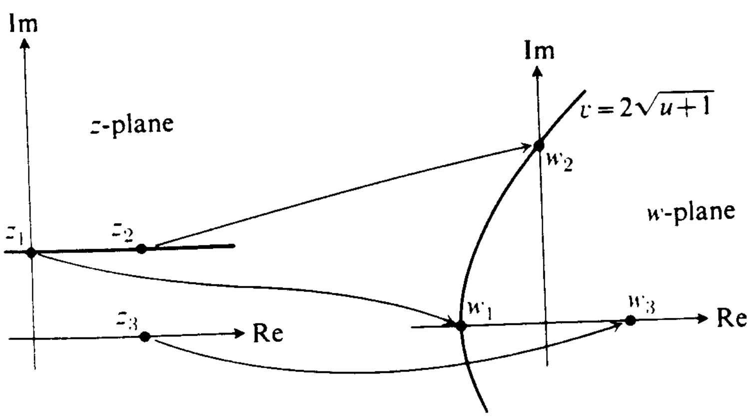

Figure 2.4

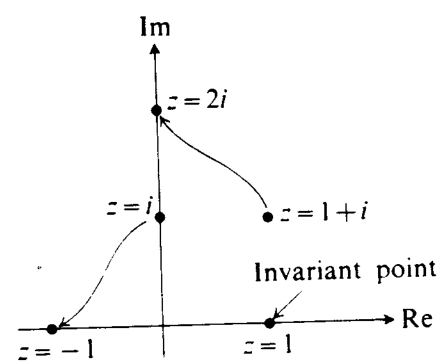

Figure 2.5

Graphical representation of complex functions poses a problem since we must deal with *four* real variables simultaneously. The idea of *mapping* is most commonly used. Two separate complex planes, the z-plane and the w-plane, are considered side by side, and a point z_0 is said to be mapped onto the point $w_0 = f(z_0)$. For instance, formula $w = z^2$ maps $z_1 = i$ onto $w_1 = i^2 = -1$; it also maps $z_2 = 1 + i$ onto $w_2 = 2i$, $z_3 = 1$ onto $w_3 = 1$, and so on. This is illustrated in Fig. 2.4, where it is also indicated that the horizontal line $y = 1$ in the z-plane is mapped onto the parabola $v = 2\sqrt{u + 1}$ in the w-plane. Sometimes it is convenient to superimpose the two planes. Then the images of various points are located on the same plane and the function $w = f(z)$ is said to transform the complex plane into itself (or a part of itself), as in Fig. 2.5, for the same function $w = z^2$.

Exercise. Show that the function $w = iz$ represents counterclockwise rotation of the complex plane by 90°. How would you describe a rotation by 180°? How would you describe a clockwise rotation by 90°?

Algebraic functions of a complex variable are defined by algebraic operations which are directly applicable to complex numbers. Transcendental functions, however, may require special definitions. Consider, for instance, the exponential function e^x (real x). Its basic properties are

$$1.\quad e^{x_1+x_2} = e^{x_1}e^{x_2}, \qquad 2.\quad (e^x)^a = e^{ax}.$$

It is desired to define a complex exponential function e^z with the same properties. Write $z = x + iy$; then

$$e^z = e^{x+iy} = e^x e^{iy}.$$

The quantity e^x is a well-defined real number, but how shall we define e^{iy}? One possible method is as follows: *Assume* that e^{iy} can be represented by the usual power series

$$e^{iy} = 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots$$

Then, rearranging the terms, we have

$$\begin{aligned} e^{iy} &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y. \end{aligned}$$

The validity of this procedure can be established after the development of the theory of convergence for complex series. However, at this stage we may simply *define* the function e^{iy} by means of

$$e^{iy} = \cos y + i \sin y.$$

This is *Euler's formula.* The desired properties,

$$e^{i(y_1+y_2)} = e^{iy_1}e^{iy_2}, \qquad (e^{iy})^n = e^{iny} \qquad (n = \text{integer}),$$

follow from the identities

$$(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) = \cos (y_1 + y_2) + i \sin (y_1 + y_2)$$

and

$$(\cos y + i \sin y)^n = \cos ny + i \sin ny.$$

The definition of a complex exponential function is then given by the formula

$$e^z = e^x(\cos y + i \sin y)$$

which has the desired properties and reduces to the real exponential function if $\operatorname{Im} z = 0$.

2.5 APPLICATIONS OF EULER'S FORMULA

Euler's formula leads to the compact *polar representation* of complex numbers,

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Suppose that a complex number z is multiplied by $e^{i\alpha}$, where α is a real constant. Then

$$e^{i\alpha}z = re^{i(\theta+\alpha)}.$$

The new number can be obtained by rotating the point z about the origin by an angle α. This fact has many important applications.

Euler's formula also permits the description of sinusoidally varying real quantities by means of complex exponentials. A general form of such quantity is

$$f(t) = a\cos(\omega t - \theta),$$

where a (amplitude), ω (angular frequency), and θ (phase) are constants, and t is a real variable (usually time). Consider the *complex* function of the *real* variable

$$g(t) = Be^{-i\omega t}$$

where B is a complex constant. Set $B = ae^{i\theta}$; then

$$\begin{aligned} g(t) = ae^{i\theta}e^{-i\omega t} &= a\cos(\theta - \omega t) + ia\sin(\theta - \omega t) \\ &= a\cos(\omega t - \theta) - ia\sin(\omega t - \theta). \end{aligned}$$

In other words, $f(t) = \mathrm{Re}\,\{g(t)\}$.

Complex functions of a real variable can be treated by the methods of calculus of real variables. For instance, if

$$g(t) = u(t) + iv(t) \qquad (u, v = \text{real functions}),$$

then

$$\frac{dg}{dt} = \frac{du}{dt} + i\frac{dv}{dt},$$

and so on. Differentiation of $Be^{-i\omega t}$ is very simple:

$$\frac{d}{dt}(Be^{-i\omega t}) = -i\omega Be^{-i\omega t}.$$

The use of complex exponentials is illustrated in the following example. Consider a (damped) harmonic oscillator subject to a harmonically varying external force. The differential equation to be solved reads

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = F\cos(\omega t - \phi) \qquad (\dot{x} = (dx/dt)\text{ etc.}),$$

where the constants α, ω_0, F, ω, and ϕ are real, and both variables x and t are real.

2.6 MULTIVALUED FUNCTIONS AND RIEMANN SURFACES

Certain complex functions are multivalued and they are usually considered as consisting of *branches*, each branch being a single-valued function of z. For instance, $f(z) = \sqrt{z}$ can be split into two branches according to the usual formula for the roots ($z = re^{i\theta}$):

1. *Principal branch*, $\quad f_1(z) = \sqrt{r}\, e^{i(\theta/2)}$,
2. *Second branch*, $\quad f_2(z) = \sqrt{r}\, e^{i[(\theta+2\pi)/2]}$.

Strictly speaking, $f_1(z)$ and $f_2(z)$ are two separate functions but they are intimately connected and for this reason they are treated together as two branches of a (double-valued) function $f(z) = \sqrt{z}$.

Note that the principal branch does not map the z-plane onto the entire w-plane, but rather onto the right half-plane ($\text{Re}\, w > 0$) to which the positive imaginary semiaxis is added. The negative imaginary semiaxis is not included. The second branch, which has no special name, maps the z-plane onto the left half-plane ($\text{Re}\, w < 0$) plus the negative imaginary semiaxis. Except for $z = 0$, no other point on the w-plane (*image plane*) is duplicated by both mappings.

Also observe another important feature of the two branches. Each branch taken separately is *discontinuous* on the negative real semiaxis. The meaning of this is as follows: The points

$$z_1 = e^{i(\pi-\delta)} \qquad \text{and} \qquad z_2 = e^{i(-\pi+\delta)},$$

where δ is a small positive number, are very close to each other. However, their images under the principal branch mapping, namely

$$f_1(z_1) = e^{i(\pi/2-\delta/2)} \qquad \text{and} \qquad f_1(z_2) = e^{-i(\pi/2-\delta/2)},$$

are very far from each other. On the other hand, note that the image of z_2 under the mapping $f_2(z)$, namely,

$$f_2(z_2) = e^{i(\pi/2+\delta/2)},$$

is very close to the point $f_1(z_1)$. It appears that the continuity of mapping can be preserved if we switch branches as we cross the negative real semiaxis.

To give this idea a more precise meaning we must define the concept of *continuous function* of a complex variable. Let $w = f(z)$ be defined in some neighborhood (see pp. 47 and 16) of point z_0 and let $f(z_0) = w_0$. We say that $f(z)$ is continuous at z_0 if* $f(z) \to w_0$ whenever $z \to z_0$ in the sense that given $\delta > 0$ (arbitrarily small), the inequality $|f(z) - w_0| < \delta$ holds whenever $|z - z_0| < \epsilon$ holds, for sufficiently small ϵ. It is readily shown† that if $w = u(x, y) + iv(x, y)$, then the continuity of w implies the continuity of $u(x, y)$ and $v(x, y)$ and vice versa.

* Also written as $\lim_{z\to z_0} f(z) = f(z_0)$.

† For example, see Kaplan, p. 495.

Riemann proposed an ingenious device to represent both branches by means of a single continuous mapping: Imagine two separate z-planes cut along the negative real semiaxis from "minus infinity" to zero. Imagine that the planes are superimposed on each other but retain their separate identity in the manner of two sheets of paper laid on top of each other. Now suppose that the second quadrant of the upper sheet is joined along the cut to the fourth quadrant of the lower sheet to form a continuous surface (Fig. 2.6). It is now possible to start a curve C in the third quadrant of the upper sheet, go around the origin, and cross the negative real semiaxis into the third quadrant of the lower sheet in a continuous motion (remaining on the surface). The curve can be continued on the lower sheet around the origin into the second quadrant of the lower sheet.

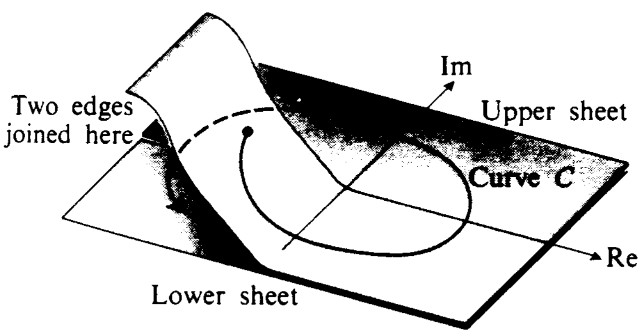

Figure 2.6

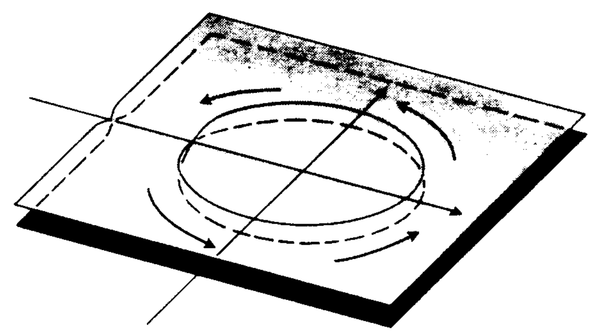

Figure 2.7

Now imagine the second quadrant of the lower sheet joined to the third quadrant of the upper sheet *along the same* cut (independently of the first joint and actually disregarding its existence). The curve C can then be continued onto the upper sheet and may *return to the starting point*. This process of cutting and cross-joining two planes leads to the formation of a *Riemann surface* which is *thought of* as a single continuous surface formed of two *Riemann sheets* (Fig. 2.7).

An important remark is now in order: The line between the second quadrant of the upper sheet and the third quadrant of the lower sheet is to be considered as *distinct* from the line between the second quadrant of the lower sheet and the third quadrant of the upper one. This is where the paper model fails us. According to this model the negative real semiaxis appears as the line where all four edges of our cuts meet. However, the Riemann surface has no such property; there are *two* real negative semiaxes on the Riemann surface just as there are two real positive semiaxes. The mapping $f(z) = \sqrt{z}$ may help to visualize this: The principal branch maps the upper Riemann sheet (negative real semiaxis excluded) onto the region Re $w > 0$ of the w-plane. The line joining the second upper with the third lower quadrants is also mapped by the principal branch onto the positive imaginary semiaxis. The lower Riemann sheet (negative real semiaxis excluded) is mapped by the second branch onto the region Re $w < 0$. The line joining the second lower with the third upper quadrants is mapped (by the second branch) onto the negative imaginary semiaxis. In this fashion the entire Riemann surface is mapped one-to-one onto the w-plane ($z = 0$ is mapped onto $w = 0$; this particular correspondence, strictly speaking, belongs to neither branch since the polar angle θ is not defined for $z = 0$).

The splitting of a multivalued function into branches is arbitrary to a great extent. For instance, define the following two functions which also may be treated as branches of $f(z) = \sqrt{z}$:

$$\textit{Branch } A\text{:}\quad f_A(z) = \begin{cases} \sqrt{r}\, e^{i(\theta/2)} & \text{for } 0 < \theta \le \pi, \\ \sqrt{r}\, e^{i[(\theta+2\pi)/2]} & \text{for } -\pi < \theta \le 0. \end{cases}$$

$$\textit{Branch } B\text{:}\quad f_B(z) = \begin{cases} \sqrt{r}\, e^{i[(\theta+2\pi)/2]} & \text{for } 0 < \theta \le \pi, \\ \sqrt{r}\, e^{i(\theta/2)} & \text{for } -\pi < \theta \le 0. \end{cases}$$

Note that branch A is continuous on the negative real semiaxis but is discontinuous on the positive real semiaxis (so is branch B). These two branches constitute, together, the double-valued function $f(z) = \sqrt{z}$, and this representation is no better and no worse than the previous one. Also observe that the Riemann surface built up by these two branches is the same as the one described before.

It is not difficult to see that the function $f(z) = \sqrt{z}$ can be split in two branches in many other ways. In all of them, however, there will be a *branch line* (or *branch cut*) extending from $z = 0$ to infinity. This line may be a curve. The Riemann surface can be obtained by joining two Riemann sheets across the cut, and this surface is unique. The point $z = 0$ where any branch line must start (or end) is called a *branch point*. The position of the branch point is determined by the nature of the multivalued function and is independent of the choice of branches.

This technique can be extended to other multivalued functions. Some require more than two Riemann sheets (for instance $f(z) = \sqrt[3]{z}$ requires three). Some require two Riemann sheets but two branch points* ($f(z) = \sqrt{(z-1)(z+1)}$), etc. There are functions requiring an infinite number of Riemann sheets such as $f(z) = z^\alpha$ with irrational α and some of transcendental functions which we shall briefly consider below.

Using the definition of exponential function,

$$e^z = e^x(\cos y + i \sin y),$$

we may define trigonometric and hyperbolic functions:

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}),$$

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{1}{\tan z},$$

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z}),$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{1}{\tanh z}.$$

* If the so-called *point at infinity* (Section 2.14) is taken into account, then the mapping $f(z) = \sqrt{z}$ also has two branch points.

All these functions are periodic: $\sin z$ and $\cos z$ have a (primitive) period 2π, $\tan z$ has a (primitive) period π, e^z, $\sinh z$, and $\cosh z$ have a (primitive) period $2\pi i$. A score of familiar formulas can be established, for instance,

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$

$$\sin z_1 - \sin z_2 = 2\cos\frac{z_1 + z_2}{2}\sin\frac{z_1 + z_2}{2}, \quad \text{etc.}$$

Also note that

$$\cosh z = \cos(iz) \qquad \sinh z = -i\sin(iz).$$

It is worthwhile to mention that $|\sin z|$ and $|\cos z|$ are by no means bounded by unity, for instance,

$$|\sin 2i| \cong 3.24.$$

The logarithmic function is defined as the inverse of exponential function. Solving $e^w = z = re^{i\theta}$ for w, we obtain the general solution

$$w = \log r + i\theta + i2n\pi \qquad (n = \text{integer}).$$

This function is multivalued: Its principal branch is usually denoted by $w = \log z$ and is defined as

$$\log z = \log r + i\theta \qquad (-\pi < \theta \leq \pi).$$

The entire multivalued function is referred to as

$$w = \operatorname{Log} z = \log z + i2n\pi.$$

These formulas are often written with the help of the *argument of* z function which is also multivalued, the principal branch being

$$\arg z = \theta \qquad (-\pi < \theta \leq \pi)$$

and the entire function reading $\operatorname{Arg} z = \arg z + 2n\pi$. Thus we may write

$$\log z = \log|z| + i\arg z, \qquad \operatorname{Log} z = \log|z| + i\operatorname{Arg} z.$$

The functions $\operatorname{Arg} z$ and $\operatorname{Log} z$ require a Riemann surface consisting of infinitely many Riemann sheets.

The definition of inverse trigonometric and hyperbolic functions now easily follows. All are multivalued:

$$\operatorname{Arc\,cos} z = i\operatorname{Log}(z + \sqrt{z^2 - 1}),$$

$$\operatorname{Arc\,sin} z = \frac{\pi}{2} - \operatorname{Arc\,cos} z,$$

$$\operatorname{Arc\,tan} z = \frac{1}{2i}\operatorname{Log}\frac{i - z}{i + z},$$

$$\operatorname{Arsinh} z = \operatorname{Log}(z + \sqrt{z^2 + 1}),$$

$$\operatorname{Arcosh} z = \operatorname{Log}(z + \sqrt{z^2 - 1}),$$

$$\operatorname{Artanh} z = \tfrac{1}{2}\operatorname{Log}\frac{1 + z}{1 - z}.$$