Differential Equations and Linear Superposition

- Basic Idea: Provide solution in closed form

- Like Integration, no general solutions in closed form

- *Order* of equation: highest derivative in equation

    e.g.

$$\frac{d^3y}{dx^3} + x\sqrt{\frac{dy}{dx}} + x^2y = 0$$

    is a 3rd order, non-linear equation.

- First Order Equations: (separable, exact, linear, tricks)

- A separable equation can be written:

$$A(x) = B(y)\frac{dy}{dx} \implies A(x)\,dx = B(y)\,dy$$

$$\int A(x)\,dx = \int B(y)\,dy + C$$

    e.g.

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$$

$$\frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}} \implies x\sqrt{1-y^2} + y\sqrt{1-x^2} + C = 0$$

- More Generally, an equation can be an *exact* differential:

$$A(x, y)\, dx + B(x, y)\, dy = 0$$

  if the left side above is the exact differential of some u(x,y) then the solution is:

$$u(x, y) = C$$

- A necessary and sufficient condition for this is:

$$\frac{\partial}{\partial y} A(x, y) = \frac{\partial}{\partial x} B(x, y)$$

  e.g.

$$(x^2 + 2y) = (-2x + y^2)\, y'$$

  is exact. We can then *partially* integrate A(x,y) in x and B(x,y) in y to guess a from for u(x,y):

$$\int (x^2 + 2y)\, dx = \frac{x^3}{3} + 2xy + C(y)$$

$$\int (2x - y^2)\, dy = 2xy - \frac{y^3}{3} + C(x)$$

$$u(x, y) = \frac{x^3 - y^3}{3} + 2xy + C$$

- For first order equations, it is always possible to find a factor $\lambda(x,y)$ which makes the equation exact. Unfortunately, there is no general technique to find $\lambda(x,y)$...

- Consider the general linear first-order equation:

$$\frac{dy}{dx} + f(x)\,y = g(x)$$

Lets try to find an integrating factor for this equation... $\lambda(x, y)$

We want $\lambda(x)\,dy + \lambda(x)\,(f(x)\,y - g(x))\,dx = 0$ to be exact. Equating the partial derivatives:

$$\frac{\partial}{\partial y}A(x, y) = \frac{\partial}{\partial x}B(x, y) \Rightarrow \frac{d}{dx}\lambda(x) = \lambda(x)f(x)$$

This new equation for lambda is separable with solution:

$$\lambda(x) = e^{\int f(x)\,dx}$$

So we can always find an integrating factor for a first order linear differential equation, provided that we can integrate f(x). Then we can solve the original equation since it is now exact.

e.g.

$$xy' + (1 + x)\,y = e^x$$

$$y' + y\frac{(x + 1)}{x} = \frac{e^x}{x}$$

The integrating factor is:

$$\lambda(x) = e^{\int f(x)\,dx} = xe^x$$

So we get the exact equation:

$$xe^x y' + e^x y\,(x + 1) = e^{2x}$$

Integrating both sides:

$$xe^x y = \frac{e^{2x}}{2} + C \Rightarrow y = \frac{e^x}{2x} + \frac{Ce^{-x}}{x}$$

- Why do we call the equation below *linear*?

$$\frac{dy}{dx} + f(x)\,y = 0$$

Consider a set of solutions:

$$y = \{y_1(x), y_2(x), \ldots\}$$

Any linear combination of the above solutions is another valid solution of the original equation. If we set:

$$y = ay_1(x) + by_2(x)$$

we get:

$$a\frac{dy_1}{dx} + b\frac{dy_2}{dx} + af(x)\,y_1(x) + bf(x)\,y_2(x) = 0$$

which is clearly satisfied if y_1 and y_2 are indeed solutions.

- Linear equations are satisfied by any linear superposition of solutions.

We divide the set of solutions into a set of linearly independent solutions satisfying the linear operator, and a particular solution satisfying the *forcing* function g(x).

In general, it can be shown that over a continuous interval, an equation of order k will have k linearly independent solutions to the homogenous equation (the linear operator), and one or more particular solutions satisfying the general (inhomogeneous) equation. A general solution is the superposition of a linear combination of homogenous solutions and a particular solution.

- A wide variety of equations of interest to an electrical engineer are in fact linear...

• Consider the circuit below:
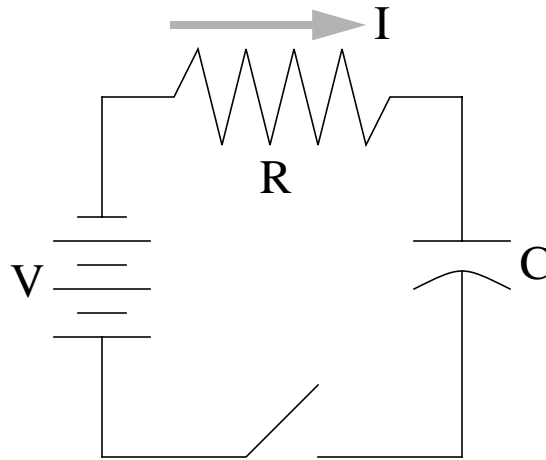
Assuming that C is initially uncharged, when the switch is closed, a current flows across the resistor, charging the capacitor. We know that the total charge Q across a capacitor is just C times the voltage across it:

$$Q(t) = CV_C(t)$$

Also, the charge on the capacitor must be the integral of the current I through the resistor:

$$Q(t) = \int I_C(t)\, dt$$

The current through the resistor is determined by the voltage across it:

$$I_R(t) = \frac{V_R(t)}{R}$$

From Kirchoff's laws, we have that the total voltage drop around a loop is zero and that the total current at each node is zero. So we can find V_C from the constant battery voltage V and the voltage across the resistor:

$$V = V_R(t) + V_C(t),\, I_R(t) = I_C(t)$$

So we get:

$$V_C(t) = \frac{1}{C}\int \frac{(V - V_C(t))}{R}\, dt$$

$$\frac{d}{dt}V_C(t) = \frac{1}{RC}(V - V_C(t))$$

Given that $V_C(t=0) = 0$, find the voltage across the capacitor as a function of time. We have:

$$\frac{d}{dt}V_C(t) + \frac{1}{RC}V_C(t) = \frac{V}{RC}$$

The homogenous solution is separable:

$$\frac{d}{dt}V_C(t) + \frac{1}{RC}V_C(t) = 0 \Rightarrow V_C(t) = e^{-\frac{t}{RC}}$$

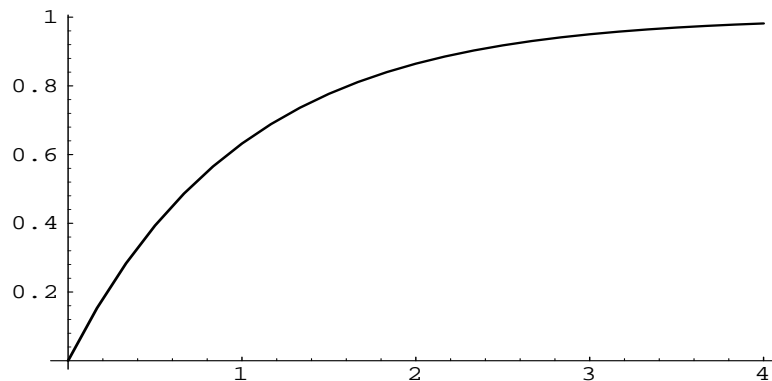And the particular solution is trivial since d/dt (V) = 0:

$$V_C(t) = V$$

So the general solution is:

$$V_C(t) = V + ae^{-\frac{t}{RC}}$$

Since $V_C(t=0) = 0$, we have:

$$V_C(t) = V - Ve^{-\frac{t}{RC}} = V\left(1 - e^{-\frac{t}{RC}}\right)$$

It is customary to set to time scale of such a circuit by the quantity RC which has the units of time. At $t = RC$, $V_C = V(1-e^{-1}) = 0.632V$.

Now consider the case in which the driving voltage is a sine wave such as power grid distribution current. We then have:

$$\frac{d}{dt}V_C(t) + \frac{1}{RC}V_C(t) = \frac{A\sin(wt)}{RC}$$

This particular solution is a bit harder than the last, but the equation still falls into the first-order linear mold: f(t) = 1/RC and g(t) = A/RC*Sin(wt). The integrating factor is:

$$\lambda(t) = e^{\frac{t}{RC}}$$

so the equation below should be exact:

$$e^{\frac{t}{RC}}\frac{d}{dt}V_C(t) + \frac{e^{\frac{t}{RC}}}{RC}V_C(t) = \frac{A\sin(wt)\,e^{\frac{t}{RC}}}{RC}$$

Carrying through the integration as before:

$$V_C = \frac{A}{RC}\frac{\frac{1}{RC}\sin(wt) - w\cos(wt)}{\frac{1}{R^2C^2} + w^2} = A\frac{(\frac{1}{RC}\sin(wt) - w\cos(wt))}{\frac{1}{RC} + w^2RC}$$

is the particular solution. Since we can superpose solutions, if we have a constant voltage and a sinewave drive, the general solution is:

$$V_C(t) = ae^{-\frac{t}{RC}} + A\frac{(\frac{1}{RC}\sin(wt) - w\cos(wt))}{\frac{1}{RC} + w^2RC}$$

An example of such a solution is the circuit with a constant voltage and a small amount of AC noise. Although the solution above looks daunting, remember that any linear combination of sin and cos functions of the same frequency can always be written in the form: Asin(wt+θ). Thus the capacitor is changing the phase of the impressed voltage. The morass of algebra above is why we often choose to write such functions as the real part of $e^{i(wt+\theta)}$. (Phasor notation).

• Example

If the "Battery" was in fact a source of alternating current, with unit frequency and amplitude, and if the inital charge on the capacitor was zero, we get the following plot for the voltage across the capacitor:
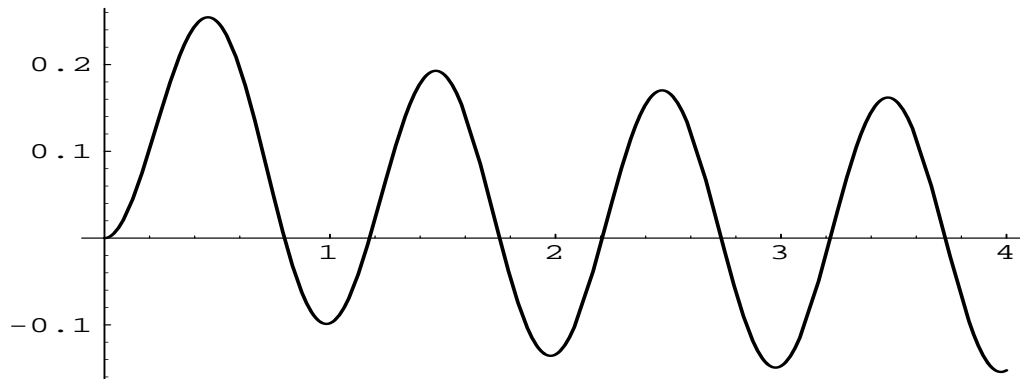
Note that there is a transient adjustment to the phase and potential. At twice the frequency, the phase and the amplitued are both changed:
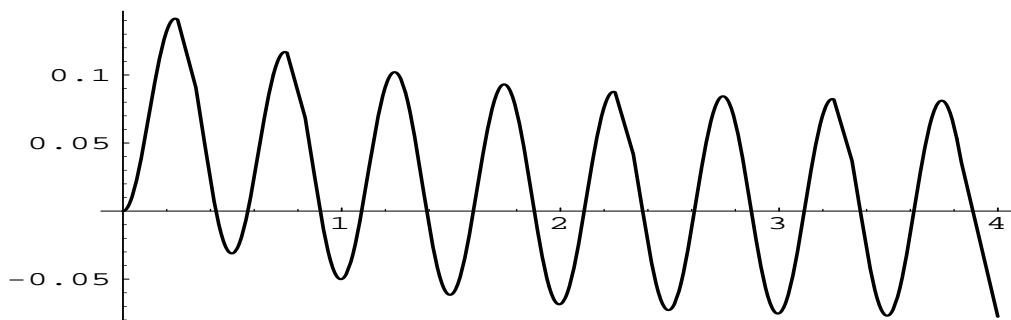
However, note that the time scale of the transient is unchanged.

• We can extend this notion of superposition of solutions to the case of multiple sources and multiple simultaneous equations.

To solve, we simply construct solutions for each of the sources acting independently, and then superpose the (add) them to find the net behavior.

For example, if the input to the previous circuit was two waves, one at unit frequency and zero phase and another at twice the frequency and zero phase, we get the following superposed solution:

$$V_C(t) \;=\; \frac{2\pi}{1+4\pi^2}e^{-t} + \frac{\sin(2\pi t) - 2\pi\cos(2\pi t)}{1+4\pi^2} \;+$$

$$\frac{4\pi}{1+16\pi^2}e^{-t} + \frac{\sin(2\pi t) - 2\pi\cos(2\pi t)}{1+4\pi^2}$$

Note, that the area of the wave tends to balance across the capacitor:
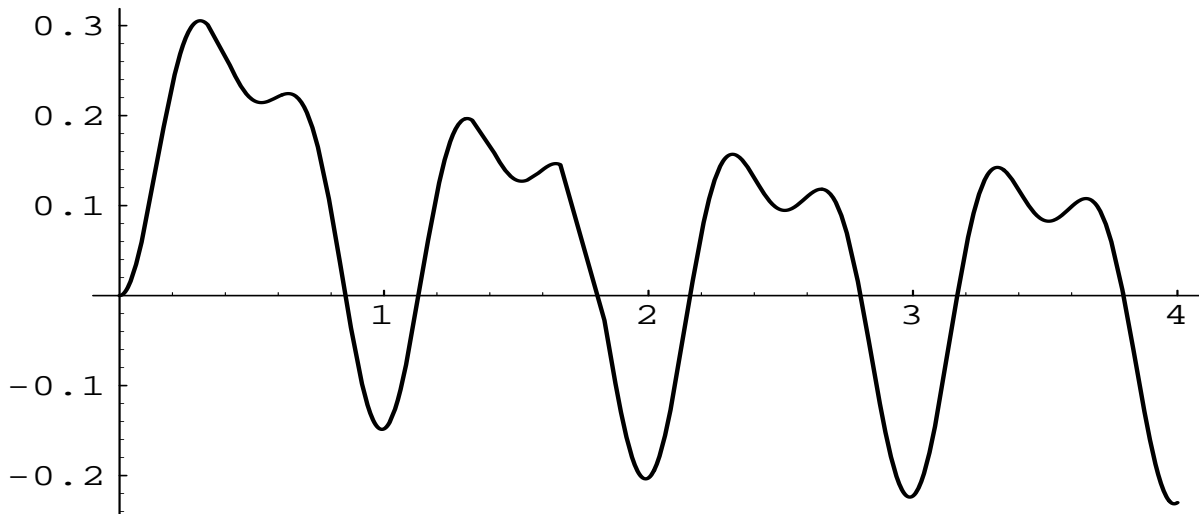
Phasors

• Represent an alternative current by a single complex number

• Trick -- use $Ae^{i\varphi}$ to represent both the amplitude and phase

> The idea here is that it is much easier to manipulate polar form complex numbers than trigonometric functions. We shall represent:

$$A\cos(\omega t + \varphi) \equiv \text{Re}(Ae^{i\varphi}e^{iwt})$$

> So far -- it doesn't look much better. However, in some applications all parts of the circuit have the same frequency of operation. In this case, we only need to carry the complex phasor $Ae^{i\varphi}$ to keep track of both.

• A phasor of the sum of two alternating signals is the same as the sum of the phasors

> So if we have two voltage sources A1 and A2, represented by $A_1 e^{i\varphi_1}$ and by $A_2 e^{i\varphi_2}$ their sum is just $A_1 e^{i\varphi_1} + A_2 e^{i\varphi_2}$. (Note that this doe not hold for the product of two phasors. It is therefore safe to represent solutions to linear differential equations as phasors and do calculations in shortand, as long as one realizes that the real-part is the desired solution.

• A second application will be the calculation of branch voltages and currents in alternating current (A.C.) circuits.

> In a general network of resistors, inductors and capacitors, the general behavior can be shown to be linear. Thus, the voltage between any two nodes or the current along any branch will have the same frequency as the source. Thus only two numbers, (a phasor) sufices to determine the circuit behavior--- the amplitude and the phase. (We shall come back to this topic in a later lecture).

Oscillating (Resonant) Circuits

• Consider the circuit below:
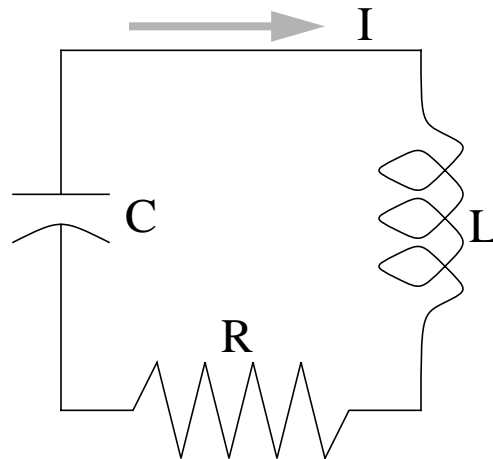
• If we take positive current as shown, we have:

$$I = -\frac{dQ}{dt} \qquad\qquad Q = CV \qquad\qquad V = L\frac{dI}{dt} + RI$$

To solve this, we will eliminate both Q and I -- to get a differential equation in V:

$$I = -\frac{d}{dt}(CV) = -C\frac{dV}{dt}$$

$$\frac{dI}{dt} = -C\frac{d^2V}{dt^2} \Rightarrow V = -LC\frac{d^2V}{dt^2} - RC\frac{dV}{dt}$$

$$\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LC}V = 0$$

This is a linear differential equation of second order (note that solve for I would also have made a second order equation!). Such equations have two indepedent solutions, and a general solution is just a superposition of the two solutions. We shall assume that initally, the voltage sotred in C is A.

Suppose we try: $V = \hat{A}e^{iwt}$ for a complex phasor $\hat{A}$. We have:

$$-\omega^2\hat{A}e^{iwt} + i\omega\frac{R}{L}\hat{A}e^{iwt} + \frac{1}{LC}\hat{A}e^{iwt} = 0$$

This equation holds for all t, w, A only if the following quadratic equation is identically zero:

$$\omega^2 - i\frac{R}{L}\omega - \frac{1}{LC} = 0$$

We get:

$$\omega = i\frac{R}{2L} \pm \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

So w is in general complex. This equation has several special cases:

• R = 0 (lossless case) then w is pure real and the solution is an alternating voltage of frequency: $\sqrt{\frac{1}{LC}}$.

$$V = Re\,(\hat{A}e^{iwt}) = A\cos\left(\frac{t}{\sqrt{LC}} + \varphi\right)$$

Here the phase is zero for the initial conditions.

• R positive, but w still has a real part:

$$\frac{1}{LC} > \frac{R^2}{4L^2} \Rightarrow R < 2\sqrt{\frac{L}{C}}$$

$$V = Ae^{-\frac{Rt}{2L}}\cos\left(t\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} + \varphi\right)$$

In this case, the real part of w defines the frequency of oscillation, while the

imaginary part determines the energy loss. Note that the amplitude decreases exponentially for increasing time.

• R large -- w is pure imaginary:

$$V = A \left(e^{-t \left(\frac{R}{2L} + \sqrt{\frac{R^2}{2L^2} - \frac{1}{LC}} \right)} + e^{-t \left(\frac{R}{2L} - \sqrt{\frac{R^2}{2L^2} - \frac{1}{LC}} \right)} \right)$$

In this case, the voltage is not oscillatory -- in fact, it simply falls exponentially to zero.

• Finally, there is a special critical value: $R = 2\sqrt{\frac{L}{C}}$

In this case, the equation above is singluar -- or equivalently, the quadratic equation for w has a repeated root.

$$\omega = i\frac{R}{2L} = \frac{i}{\sqrt{LC}}$$

$$V = (A + Bt) e^{-\frac{t}{\sqrt{LC}}}$$

• In each of the above cases, we have consistantly used the notion that the value was represented by the real part of the complex polar form. This trick has saved an enormous amount of work -- compared to using trigonometric functions.

More Techniques for Differential Equations

• A particularly powerful technique is that of making an appro-
priate change of variable in an equation.

> For example,
>
> $$y' = f(ax + by + c)$$
>
> is easily solved if we make the substitution: u = ax + by + c

• A variable change is performed by writing the form to be sub-
stituted, find each (total) derivative and then substituting.

> e.g.
>
> $$u = ax + by + c, u' = a + by'$$
>
> So the equation above becomes:
>
> $$\frac{u' - a}{b} = f(u) \iff \frac{du}{a + bf(u)} = dx$$
>
> which is separable. Note that after the integration, we must replace u by
> ab+by+c to get the final solution.

• In general, a appropriate substitution is a very powerful tech-
nique to simplify or solve a typical differential equation. Recog-
nizing that a particular substitution is useful is often a tricky
proposition. Some special cases are listed below:

• $y' + f(x) y = g(x) y^n$ "Bernoulli Equation" can be made linear
by setting $u = y^{1-n}$.

> Note that:
>
> $$u' = (1 - n) y^{-n} y'$$

and:

$$\frac{y'}{y^n} + \frac{f(x)}{y^{n-1}} = g(x)$$

So substituting:

$$\frac{u'}{1-n} + uf(x) = g(x)$$

which is linear.

• Homogenous first-order equations:

$$A(ax, ay) = a^r A(x, y)$$

denotes a homogenous function of degree r. A general first order equation is homogenous if both A and B are homogenous functions. Then

$$A(x, y)\, dx + B(x, y)\, dy = 0$$

can be solved by substituting:

$$y(x) = xu(x),\, dy = u(x)\, dx + x\, du$$

E.g.

$$y\, dx + (2\sqrt{xy} - x)\, dy = 0$$

Here r = 1. If we substitute:

$$xu(x)\, dx + (2\sqrt{x^2 u} - x)(u(x)\, dx + x\, du) = 2u^{\frac{3}{2}}\, dx + (2\sqrt{u} - 1)\, x\, du = 0$$

$$\frac{2dx}{x} = \frac{(2\sqrt{u} - 1)}{u^{3/2}}\, du$$

which is separable.

• If the dependent variable y is missing, substitute y'=u, lowering the order by one.

e.g.

$$y'' + f(x)\,y' + g(x) \;=\; 0 \Rightarrow u' + f(x)\,u + g(x) \;=\; 0$$

Then solve and integrate when done...

• If the independent variable is missing, let y be the new independent variable, and let u = y', u' = y'' etc. This also lowers the order by one.

e.g.

$$y'' + f(y') + g(y) \;=\; 0 \Rightarrow u' + f(u) + g(y) \;=\; 0$$

e.e.g.

$$y'' + y' + y^2 \;=\; 0 \Rightarrow u' + u + y^2 \;=\; 0$$

The homogenous solution:

$$u' + u \;=\; 0 \Rightarrow u \;=\; ae^{-y}$$

The particular solution:

$$u \;=\; -y^2 + 2y - 2$$

In general:

$$u \;=\; y' \;=\; ae^{-y} - y^2 + 2y - 2$$

So finally:

$$x \;=\; \int \frac{1}{ae^{-y} - y^2 + 2y - 2}\,dy$$

- Always watch for an exact equation or for an appropriate integrating factor:

  e.g.

  $$y'' = f(y)$$

  can be integrated immediately if both sides are multiplied by y'...

  $$y''y' = f(y)y' \Rightarrow \frac{1}{2}(y')^2 = \int f(y)\,dy$$

  e.e.g.

  $$y'' = y^2 \Rightarrow \frac{1}{2}(y')^2 = \frac{y^3}{3} \Rightarrow \frac{\sqrt{6}}{2}y' = \pm y^{\frac{3}{2}}$$

  $$y = \frac{6}{(x+C)^2}$$

- A particularly useful substitution is one of the simplest:

  $$y = u(x)\,v(x)$$

  for a linear second order equation. Given the equation:

  $$y'' + f(x)\,y' + g(x)\,y = 0$$

  Substituting, we obtain:

  $$v'' + \left(\frac{2u'}{u} + f(x)\right)v' + \left(\frac{u'' + fu' + gu}{u}\right)v = 0$$

  Now, it is clear that special chioces for u will potentially give us simplified equations for v... For example, if u is a solution to the original equation, the thrid term drops out, leaving an equation than can be reduced to first order, linear. If we try:

  $$u = e^{\left(-\frac{1}{2}\right)\int f(x)\,dx}$$

  then u satisfies the second term and it drops out, leaving a simpler second order equation.

e.g.

If our equation is:

$$y'' + \frac{1}{2x}y' - \frac{1}{4x}y = 0$$

and we happen to guess that one solution is:

$$y = e^{\sqrt{x}}$$

we can use the previous substitution to find the other solution... So we substitute:

$$y(x) = v(x)e^{\sqrt{x}}, \, y'(x) = \left(\frac{v(x)}{2\sqrt{x}} + v'(x)\right)e^{\sqrt{x}}$$

$$y''(x) = \left(-\frac{v}{4\sqrt{x^3}} + \frac{v}{4x} + \frac{v'}{\sqrt{x}} + v''\right)e^{\sqrt{x}}$$

And we get:

$$v'' + \frac{v'}{\sqrt{x}} + \frac{v'}{2x} = 0$$

This equation can be simplified by letting w = v':

$$w' + w\left(\frac{1}{\sqrt{x}} + \frac{1}{2x}\right) = 0$$

Which is separable-- we get:

$$\log(w) + C + 2\sqrt{x} + \frac{1}{2}\log(x) = 0$$

$$w = \frac{ae^{-2\sqrt{x}}}{\sqrt{x}} = v'$$

So integrating, one last time:

$$v = (-c)e^{-2\sqrt{x}} \Rightarrow y = (-c)e^{-2\sqrt{x}}e^{\sqrt{x}} = -ce^{-\sqrt{x}}$$

is the other solution....

• We can use the preceeding trick to help find particular solutions as well.

Consider the following 2nd order linear equation:

$$y'' + f(x)\, y' + g(x)\, y \;=\; h(x)$$

If we know a solution of the homogenous equation, u(x), we can substitute as before to get a new linear 1st order equation in v'(x). Solving this equation and then integrating again will allow the particular solution to be found. This might be an easier procedure than variation of parameters...

e.g.

$$y'' + (x^2 - 4)\, y' - 4x^2 y \;=\; 2xe^{-\frac{x^3}{3}}$$

Note that a solution to the homogenous equation is simply:

$$y \;=\; ae^{4x}$$

So we substitute:

$$y \;=\; v(x)\, e^{4x},\; y' \;=\; v'e^{4x} + 4ve^{4x},\; y'' \;=\; 16e^{4x}v + 8e^{4x}v' + e^{4x}v''$$

To obtain:

$$v'' + (x^2 + 4)\, v' \;=\; 2xe^{-\frac{x^3}{3}}$$

This equation is first order, linear in v' -- we get:

$$v'(x) \;=\; \frac{4x - 1}{8e^{\frac{x^3}{3}}} + ae^{-4x - \frac{x^3}{3}}$$