1. a) $y''' - 2y'' - 3y' = 0$

$u = y'$

$u'' - 2u' - 3u = 0$

characteristic equation

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

roots are $r = 3, -1$

$u = c_1 e^{3x} + c_2 e^{-x}$

$$\boxed{y = \int u = c_1 e^{3x} + c_2 e^{-x} + c_3}$$

b) $y'' - y' = x^2$

homogenous solution

$u = y'$

$u' - u = x^2$

$u' - u = 0$

characteristic equation

$r - 1 = 0$

root $r = 1$

$u_h = c_1 e^x$

particular solution

use a forcing function

Assume $u(x)$ is of the form

$u(x) = Ax^2 + Bx + C$

$u'(x) = 2Ax + B$

plug into: $u' - u = x^2$

$2Ax + B - Ax^2 - Bx - C = x^2$

solve for A, B, C

$-A = 1$	$A = -1$
$2A - B = 0$	$B = -2$
$B - C = 0$	$C = -2$

$u_p = -x^2 - 2x - 2$

$u = y' = c_1 e^x - x^2 - 2x - 2$

$y = \int u = \int (c_1 e^x - x^2 - 2x - 2)\,dx$

$$\boxed{y = c_1 e^x - \tfrac{1}{3}x^3 - x^2 - 2x + c_2}$$

1. c) $x^2y' + y^2 = xyy'$

$(x^2 - xy)y' + y^2 = 0$

$(x^2 - xy)dy + y^2 dx = 0$

we now have an equation of the form

$A(x,y)\,dx + B(x,y)\,dy = 0$

$A(x,y) = y^2 \quad B(x,y) = (x^2 + xy)$

✓ to make sure

A & B are homogenous

$A(ax,ay) = (ay)^2 = a^2y^2 = a^2 A(x,y)$

$B(ax,ay) = ((ax)^2 - (ax)(ay)) = a^2 B(x,y)$

$\therefore$ $A(x,y)$ and $B(x,y)$ are homogenous

substitute

$y(x) = xu(x)$

$dy(x) = u(x)\,dx + x\,du$

$u(x) = \cancel{\frac{x}{y}}\ \frac{y}{x}$

$(x^2 - xy)dy + y^2 dx = 0$

$(1 - \frac{y}{x})dy + \frac{y^2}{x^2}dx = 0$

$(1-u)(u(x)dx + xdu) + u^2 dx = 0$

$udx + xdu - u^2dx - xudu + u^2dx = 0$

$udx + xdu - xudu = 0$

$udx + x(1-u)du = 0$

$x(1-u)du = -udx$

$\frac{-1}{x}dx = \frac{(1-u)}{u}du$

integrate both sides

$-\ln x = \ln u - u + C$

$-(\ln(xu)) = -u + C$

$-\ln(x\frac{y}{x}) = -y/x + C$

$\boxed{-\ln y = -y/x + C}$

2. $y''-4y'+x^2(y'-4y)=0$

$y=e^{4x}$ is a nonzero solution of the above equation

Assume the solution to the nonhomogeneous differential equation to be in the form: $y=e^{4x}\cdot\theta(x)$

$y'=4\theta e^{4x}+\theta' e^{4x}$
$y''=16\theta e^{4x}+8\theta' e^{4x}+\theta'' e^{4x}$

$y''-4y'+x^2(y'-4y)=2xe^{-x^3/3}$

$(\cancel{16\theta}+8\theta'+\theta'')e^{4x}-4(\cancel{4\theta}+\theta')e^{4x}+x^2(4\theta+\theta'-4\theta)e^{4x}=2xe^{-x^3/3}$

$8\theta'+\theta''-4\theta'+x^2\theta'=2xe^{-x^3/3-4x}$

$\theta''+\theta'(8-4+x^2)=2xe^{-x^3/3-4x}=\theta''+\theta'(4+x^2)=2xe^{-x^3/3-4x}$

$\theta'=w$
$w'+w(4-x^2)=2xe^{-x^2/3-4x}$
$\lambda=e^{\int(4+x^2)dx}=e^{4x+x^3/3}$
$e^{4x+x^3/3}\cancel{dw}w'+(4+x^2)e^{(4x+x^3/3)}w=2x$ is an exact differential

$\underbrace{e^{4x+x^3/3}}_{\partial G/\partial w}dw+\underbrace{[(4+x^2)e^{4x+x^3/3}w-2x]}_{\partial G/\partial x}dx=0$

$\partial G/\partial w=e^{4x+x^3/3}\Rightarrow G=e^{(4x+x^3/3)}w+f(x)$

$\partial G/\partial x=(4+x^2)e^{4x+x^3/3}w-2x\Rightarrow G=e^{(4x+x^3/3)}w-x^2+C$

$\therefore e^{4x+x^3/3}w-x^2+C=C_2$

$e^{4x+x^3/3}w=x^2+C$

$w=\dfrac{x^2+C}{e^{4x+x^3/3}}$

$\theta=\int w=\int(x^2+c)e^{-4x-x^3/3}dx$

$=\int(x^2+4)e^{-4x-x^3/3}dx+\int(c_1-4)e^{-4x-x^3/3}dx$

$=-e^{-4x-x^3/3}+\int(c_1-4)e^{-4x-x^3/3}dx+c_2$

$y=e^{4x}\theta$
$y(0)=\theta(0)=-1+\int(c_1-4)e^{-4x-x^3/3}dx|_{x=0}+c_2=0$

$y'(0) = 4$

$y'(0) = 4 \cdot \cancelto{0}{\theta(0)} e^{4 \cdot 0} + e^{4 \cdot 0} \theta'(0) = 4$

$= \theta'(0) = \omega(0) = \dfrac{x^2 + c_1 + 4 - 4}{e^{4x + x^3/3}} = c_1 = 4$

for $c_1 = 4$

$\theta = -e^{-4x - x^3/3} + c_2$

$\theta(0) = 0 = -1 + c_2$

$\therefore c_2 = 1$

$\therefore y = (-e^{-4x + x^3/3} \cancel{} + 1) e^{4x}$

$\boxed{y = -e^{x^3/3} + e^{4x}}$

3.

$$v'' + \frac{R}{L}v' + \frac{1}{LC} = \sin(\alpha t)$$

use a forcing function of the form

$$v(t) = A\sin(\alpha t) + B\cos(\alpha t)$$

$$v'(t) = A\alpha\cos(\alpha t) - B\alpha \cancel{\cos}\,\overset{\sin}{}(\alpha t)$$

$$v''(t) = -A\alpha^2\sin(\alpha t) - B\alpha^2\cos(\alpha t)$$

Plug into equation

$$-A\alpha^2\sin(\alpha t) - B\alpha^2\cos(\alpha t) + \frac{R}{L}A\alpha\cos(\alpha t) - \frac{R}{L}B\alpha\sin(\alpha t) + \frac{1}{LC}A\sin(\alpha t)$$
$$+ \frac{1}{LC}B\cos(\alpha t) = \sin(\alpha t)$$

cos terms $= 0$

$$-B\alpha^2 + \frac{R}{L}A\alpha + \frac{1}{LC}B = 0$$

sin terms $= 1$

$$-A\alpha^2 - \frac{R}{L}B\alpha + \frac{1}{LC}A = 1$$

$$B\left(\alpha^2 - \frac{1}{LC}\right) = A\frac{R}{L}\alpha$$

$$A = \frac{\left(\alpha^2 - \frac{1}{LC}\right)B}{\frac{R}{L}\alpha}$$

$$1 = A\left(\frac{1}{LC} - \alpha^2\right) - \frac{R}{L}\alpha B$$

$$1 = \frac{-\left(\alpha^2 - \frac{1}{LC}\right)^2}{\frac{R}{L}\alpha}B - \frac{R}{L}\alpha B$$

$$= \frac{-\left(\alpha^2 - \frac{1}{LC}\right)^2 - \left(\frac{R}{L}\alpha\right)^2}{\frac{R}{L}\alpha}B$$

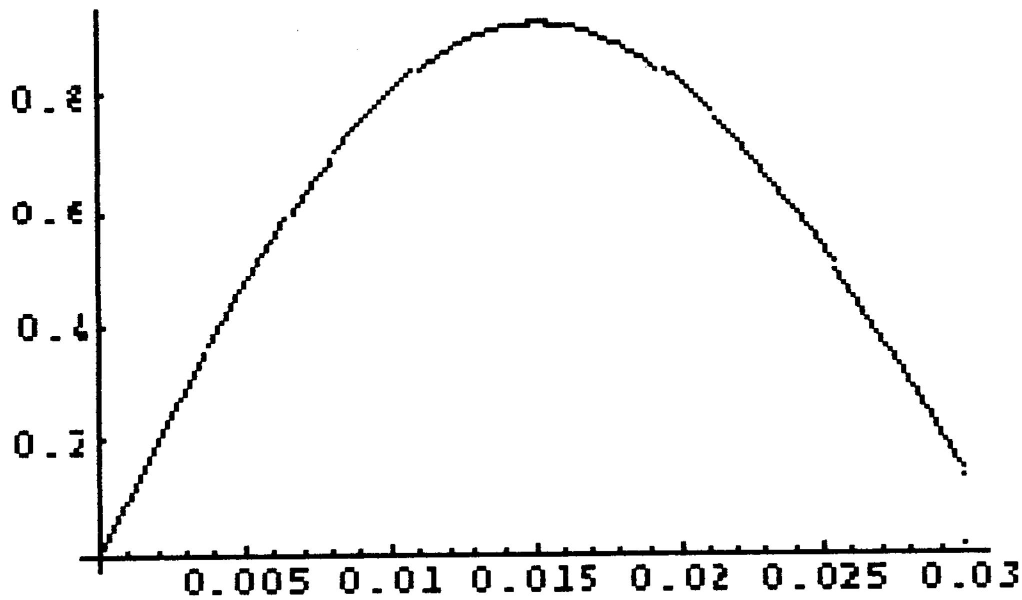
$$\boxed{\begin{aligned} B &= \frac{\frac{R}{L}\alpha}{-\left(\alpha^2 - \frac{1}{LC}\right)^2 - \left(\frac{R}{L}\alpha\right)^2} \\ A &= \frac{\left(\alpha^2 - \frac{1}{LC}\right)}{-\left(\alpha^2 - \frac{1}{LC}\right)^2 - \left(\frac{R}{L}\alpha\right)^2} \\ V_p(t) &= A\sin(\alpha t) + B\cos(\alpha t) \end{aligned}}$$

for given initial conditions:

$$v(t) \sim 1.02 \times 10^{-8}\left(e^{-5t}\cos(99.9t) - \cos(\alpha t)\right)$$
$$+ 1.001 \times 10^{0}\, e^{-5t}\sin(99.9t)$$
$$\sim 1.001\, e^{-5t}\sin(99.9t)$$

$$v(t) = V_h(t) + V_p(t)$$

36. plot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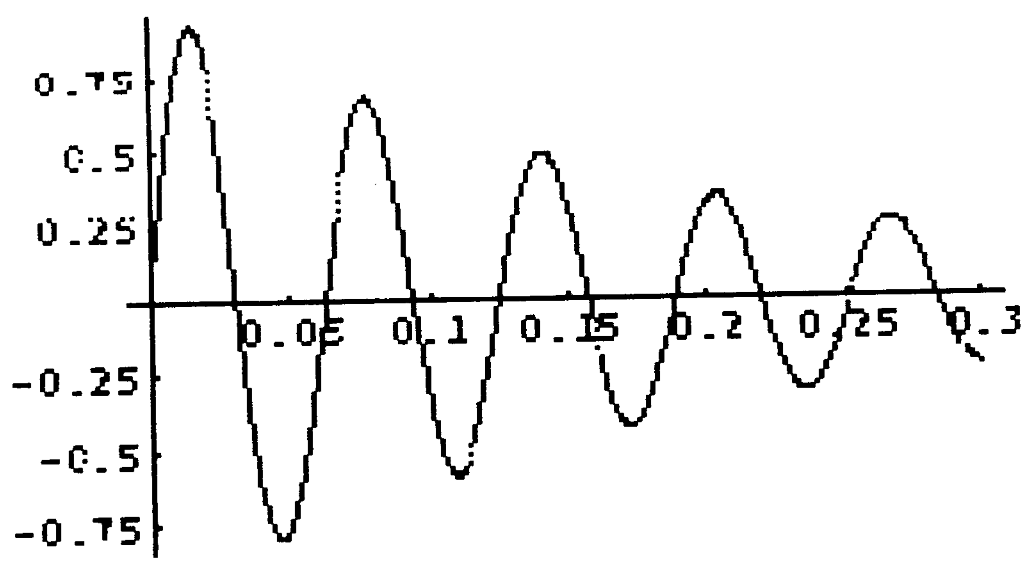
0.6
0.4
0.2
0.005 0.01 0.015 0.02 0.025 0.03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0.25
-0.25
-0.5
-0.75
0.05 0.1 0.15 0.2 0.25 0.3