Models for Concurrency: Towards a Classification

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Abstract

Models for concurrency can be classified with respect to three relevant parameters: behaviour/system, interleaving/noninterleaving, linear/branching time. When modelling a process, a choice concerning such parameters corresponds to choosing the level of abstraction of the resulting semantics.

In this paper, we move a step towards a classification of models for concurrency based on the parameters above. Formally, we choose a representative of any of the eight classes of models obtained by varying the three parameters, and we study the formal relationships between using the language of category theory.

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Introduction

Much effort in the development of the *theory of concurrency* has been devoted to the study of suitable models for concurrent and distributed processes, and to the formal understanding of their semantics.

As a result, in addition to standard models like languages, automata and transition systems [6, 13], models like *Petri nets* [12], *process algebras* [9, 4], *Hoare traces* [5], *Mazurkiewicz traces* [8], *synchronisation trees* [20] and *event structures* [10, 21] have been introduced.

The idea common to the models above is that they are based on atomic units of change—transitions, actions, events or symbols from an alphabet—which are *indivisible* and constitute the steps out of which computations are built.

The difference between the models may be expressed in terms of the parameters according to which models are often classified. For instance, a distinction made explicitly in the theory of Petri nets, but sensible in a wider context, is that between so-called *'system'* models allowing an explicit representation of the (possibly repeating) states in a system, and *'behaviour'* models abstracting away from such information, which focus instead on the behaviour in terms of patterns of occurrences of actions over time. Prime examples of the first type are transition systems and Petri nets, and of the second type, trees, event structures and traces. Thus, we can distinguish among models according to whether they are *system* models or *behaviour* models, in this sense. Further distinctions are whether they can faithfully take into account the difference between *concurrency* and *nondeterminism* and, finally, whether they can represent the *branching structure* of processes, i.e., the points in which choices are taken, or not. So, relevant parameters when looking at models for concurrency are

> *Behaviour or System model;*
> *Interleaving or Noninterleaving model;*
> *Linear or Branching Time model.*

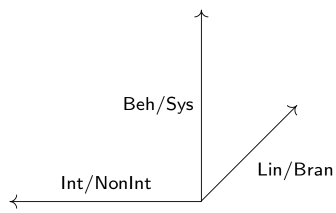
These parameters correspond to choices of the *level of abstraction* at which we examine processes and which are not necessarily fixed for a process once and for all. It is the actual application one has in mind for the formal semantics which guides the choice of the abstraction level. It can therefore be of value to be able to move back and forth between the representation of a process in one model and its representation in another, if possible in a way which respects its structure. In other words, it is relevant to study translations between models, and particularly with respect to the three parameters above.

This work presents a first step towards a classification of models for concurrency based on the three parameters, which also represent a further step towards the identification of systematic connections between transition based models. More precisely, we study a *representative* for each of the eight classes of models obtained by varying the parameters *behaviour/system*, *interleaving/noninterleaving* and *linear/branching* in all the possible ways. Intuitively, the situation can be graphically represented, as in the picture below, by a

Beh/Int/Lin	*Hoare languages*	HL
Beh/Int/Bran	*synchronisation trees*	ST
Beh/NonInt/Lin	*deterministic labelled event structures*	dLES
Beh/NonInt/Bran	*labelled event structures*	LES
Sys/Int/Lin	*deterministic transition systems*	dTS
Sys/Int/Bran	*transition systems*	TS
Sys/NonInt/Lin	*deterministic transition systems with independence*	dTSI
Sys/NonInt/Bran	*transition systems with independence*	TSI

Table 1: The models

three-dimensional frame of reference whose coordinate axes represent the three parameters.



Our choices of models are summarised in Table 1. It is worth noticing that, with the exception of the new model of *transition systems with independence*, each model is well-known.

The formal relationships between models are studied in a *categorical* setting, using the standard categorical tool of *adjunctions*. The 'translations' between models we shall consider are *coreflections* or *reflections*. These are particular kinds of adjunctions between two categories which imply that one category is *embedded*, fully and faithfully, in another.[1]

Here we draw on the experience in recasting models for concurrency as categories, detailed, e.g., in [22]. Briefly the idea is that each model (transition systems are one such model) will be equipped with a notion of morphism, making it into a category in which the operations of process calculi are universal constructions. The morphisms will preserve behaviour, at the same time respecting a choice of granularity of the atomic changes in the description of processes—they are forms of *simulations*. One of their roles is to relate the behaviour of a construction on processes to that of its components. The reflections and coreflections provide a way to express that one model is embedded in (is more abstract than) another, even when the two models are expressed in very different mathematical terms. One adjoint will say how to embed the more abstract model in the other, the other will abstract away from some aspect of the representation. The preservation properties of adjoints can be used to show how a semantics in one model *translates* to a semantics in another.

The diagram below, in which arrows represent coreflections and the 'backward' arrows reflections, shows the 'cube' of relationships summarizing the re-

[1]Here a coreflection is an adjunction in which the unit is a natural isomorphism, and a reflection an adjunction where the counit is a natural isomorphism.

sults of this paper.

$$
\begin{tikzcd}
 & \underline{\mathsf{TSI}} & & \underline{\mathsf{TS}} \arrow[ll, hook', -{Triangle[open]}] \\
\underline{\mathsf{dTSI}} \arrow[ur, hook, -{Triangle[open]}] & & \underline{\mathsf{dTS}} \arrow[ll, hook', -{Triangle[open]}] \arrow[ur, hook, -{Triangle[open]}] & \\
 & \underline{\mathsf{LES}} \arrow[uu, hook, -{Triangle[open]}] & & \underline{\mathsf{ST}} \arrow[ll, hook', -{Triangle[open]}] \arrow[uu, hook, -{Triangle[open]}] \\
\underline{\mathsf{dLES}} \arrow[uu, hook, -{Triangle[open]}] \arrow[ur, hook, -{Triangle[open]}] & & \underline{\mathsf{HL}} \arrow[ll, hook', -{Triangle[open]}] \arrow[uu, hook, -{Triangle[open]}] \arrow[ur, hook, -{Triangle[open]}] &
\end{tikzcd}
$$

Although our main concern here is conceptual, on abstract relationships between models, of course all the 'abstraction' adjoints have clear *computational* meanings and, therefore, possible applications. In particular, moving along $\mathsf{NonInt} \mapsto \mathsf{Int}$ enforces the reduction of concurrency to nondeterminism, whilst moving along $\mathsf{Sys} \mapsto \mathsf{Beh}$ is essentially moving from 'machines' to their 'behaviours'. The translations $\mathsf{Bran} \mapsto \mathsf{Lin}$ purge the models from nondeterministic branching, enforcing a linear time setting. The usefulness, e.g., in specification, verification, and semantics, of these reductions is largely proved in literature.

Establishing the coreflection $\underline{\mathsf{LES}} \hookrightarrow \underline{\mathsf{TSI}}$, the new notion of occurrence transition systems with independence arises naturally. These prove to be rather interesting structures. In particular, by means of them we shall identify yet another characterisation of *coherent*, *finitary*, *prime algebraic domains*, one expressible simply in terms of the structure of transition systems.

Although most of the chosen models are well known, among the adjunctions in the cube only $\underline{\mathsf{HL}} \hookleftarrow \underline{\mathsf{ST}}$, $\underline{\mathsf{ST}} \hookrightarrow \underline{\mathsf{TS}}$ and $\underline{\mathsf{ST}} \hookrightarrow \underline{\mathsf{LES}}$ have already appeared in literature. Some related results are presented in [2], in which the authors focus on the interleaving/noninterleaving and linear/branching axes studying the relationships between four chosen models of concurrency different from ours.

This paper is a full and extended version of [15]; some of the results presented here appear also in [11, 14]. In order to keep the size of the paper in reasonable bounds, some of the most technical proofs are only sketched.

1 Preliminaries

In this section, we study the interleaving models. We start by briefly recalling some well-known relationships between languages, trees and transition systems [22], and then, we study how they relate to deterministic transition systems.

Definition 1.1 (*Labelled Transition Systems*)
A labelled transition system is a structure $T = (S, s^I, L, \mathit{Tran})$ where S is a set of states, $s^I \in S$ is the initial state, L is a set of labels, and $\mathit{Tran} \subseteq S \times L \times S$ is the transition relation.

The fact that $(s, a, s') \in Tran_T$—also denoted by $s \xrightarrow{a} s'$, when no ambiguity is possible—indicates that the system can evolve from state s to state s' performing an action a. The structure of transition systems immediately suggests a notion of simulation morphisms: initial states must be mapped to initial states, and for every action the first system can perform in a given state, it must be possible for the second system to perform the corresponding action—if any—from the corresponding state. This guarantees that morphisms are *simulations*.

Definition 1.2 (*Labelled Transition System Morphisms*)
Given the labelled transition systems T_0 and T_1, a morphism $h: T \to T'$ is a pair (σ, λ), where $\sigma: S_{T_0} \to S_{T_1}$ is a function and $\lambda: L_{T_0} \rightharpoonup L_{T_1}$ a partial function, such that[2]

i) $\sigma(s^I_{T_0}) = s^I_{T_1}$;

ii) $(s, a, s') \in Tran_{T_0}$ *implies* $\begin{array}{ll} \big(\sigma(s), \lambda(a), \sigma(s')\big) \in Tran_{T_1}, & \textit{if } \lambda{\downarrow}a \\ \sigma(s) \ = \ \sigma(s'), & \textit{otherwise.} \end{array}$

It is immediate to see that labelled transition systems and labelled transition system morphisms, when the obvious componentwise composition of morphisms is considered, give a *category*, which will be referred to as $\underline{\mathsf{TS}}$.

A particularly interesting class of transition systems is that of *synchronisation trees*, i.e., the tree-shaped transition systems.

Definition 1.3 (*Synchronisation Trees*)
A synchronisation tree is an acyclic, reachable transition system S such that

$$(s', a, s),\ (s'', b, s) \in Tran_S \qquad \textit{implies} \qquad s' = s'' \quad \textit{and} \quad a = b$$

We shall write $\underline{\mathsf{ST}}$ to denote the full subcategory of $\underline{\mathsf{TS}}$ consisting of synchronisation trees.

In a synchronisation tree part of the information about the internal structure of systems is lost, whilst the information about their behaviour is maintained. In particular, it is not anymore possible to discriminate between a system which reaches again and again the same state, and a system which passes through a sequence of states, as far as they are able to perform the same actions. However, observe that the nondeterminism present in a state can still be expressed in full generality. In this sense, synchronisation trees are *branching time* and *interleaving* models of *behaviours*.

A natural way of studying the behaviour of a system consists of considering its computations as a synchronisation tree, or, in other words, of *'unfolding'* the transition system by decorating each state with the history of the computation which reached it.

[2] We use $f{\downarrow}x$ to mean that a partial function f is defined on argument x. Dually, $\uparrow$ stands for undefined.

Definition 1.4 (*Unfoldings of Transition Systems*)
Let T be a transition system. A path π of T is ϵ, the empty path, or a sequence $t_1 \cdots t_n$, $n \geq 1$, where

i) $t_i \in \mathit{Tran}_T$, *for* $i = 1, \ldots, n$;

ii) $t_1 = (s^I_T, a_1, s_1)$ *and* $t_i = (s_{i-1}, a_i, s_i)$, *for* $i = 2, \ldots, n$.

We shall write $\mathit{Path}(T)$ to indicate the set of paths of T and π_s to denote a generic path leading to state s.
Define $ts.st(T)$ to be the synchronisation tree $(\mathit{Path}(T), \epsilon, L_T, \mathit{Tran})$, where

$$\big((t_1 \cdots t_n), a, (t_1 \cdots t_n t_{n+1})\big) \in \mathit{Tran} \\ \Leftrightarrow \quad t_n = (s_{n-1}, a_n, s_n) \quad \text{and} \quad t_{n+1} = (s_n, a, s_{n+1}).$$

This procedure amounts to abstracting away from the internal structure of a transition system and looking at its behaviour. It is very interesting to notice that this simple construction is functorial and, moreover, that if forms the right adjoint to the inclusion functor of $\underline{\mathsf{ST}}$ in $\underline{\mathsf{TS}}$. In other words, the category of synchronisation trees is coreflective in the category of transition systems. The *counit* of such adjunction is the morphism $(\phi, id_{L_T})\colon ts.st(T) \to T$, where $\phi\colon \mathit{Path}(T) \to S_T$ is given by $\phi(\epsilon) = s^I_T$, and $\phi\big((t_1 \cdots t_n)\big) = s$ if $t_n = (s', a, s)$.

While looking at the behaviour of a system, a further step of abstraction can be achieved forgetting also the branching structure of a tree. This leads to another well-know model of behaviour: *Hoare languages.*

Definition 1.5 (*Hoare Languages*)
A Hoare language is a pair (H, L), where $\varnothing \neq H \subseteq L^*$, and $sa \in H \Rightarrow s \in H$.
A partial map $\lambda\colon L_0 \rightharpoonup L_1$ is a morphism of Hoare languages from (H_0, L_0) to (H_1, L_1) if for each $s \in H_0$ it is $\hat{\lambda}(s) \in H_1$, where $\hat{\lambda}\colon L_0^* \to L_1^*$ is defined by

$$\hat{\lambda}(\epsilon) = \epsilon \qquad \text{and} \qquad \hat{\lambda}(sa) = \begin{cases} \hat{\lambda}(s)\lambda(a) & \text{if } \lambda{\downarrow}a; \\ \hat{\lambda}(s) & \text{otherwise.} \end{cases}$$

These data give the category $\underline{\mathsf{HL}}$ of Hoare languages.

Observe that any language (H, L) can be seen as a synchronisation tree just by considering the strings of the language as states, the empty string being the initial state, and defining a transition relation where $s \xrightarrow{a} s'$ if and only if $sa = s'$. Let $hl.st\big((H, L)\big)$ denote such a synchronisation tree.

On the contrary, given a synchronisation tree S, it is immediate to see that the strings of labels on the paths of S form a Hoare language. More formally, for any transition system T and any path $\pi = (s^I_T, a_1, s_1) \cdots (s_{n-1}, a_n, s_n)$ in $\mathit{Path}(T)$, define $\mathit{Act}(\pi)$ to be the string $a_1 \cdots a_n \in L^*_T$. Moreover, let $\mathit{Act}(T)$ denote the set of strings

$$\big\{\mathit{Act}(\pi) \bigm| \pi \in \mathit{Path}(T)\big\}.$$

Then, the language associated to S is $st.hl(S) = Act(S)$, and simply by defining $st.hl\big((\sigma,\lambda)\big) = \lambda$, we obtain a functor $st.hl: \underline{\mathsf{ST}} \to \underline{\mathsf{HL}}$. Again, this constitutes the left adjoint to $hl.st: \underline{\mathsf{HL}} \to \underline{\mathsf{ST}}$ and given above. The situation is illustrated below, where $\hookrightarrow$ represents a coreflection and $\hookleftarrow$ a reflection.

Theorem 1.6

$$\underline{\mathsf{HL}} \hookleftarrow \underline{\mathsf{ST}} \hookrightarrow \underline{\mathsf{TS}}.$$

The existence of a (co)reflection from category $\underline{\mathsf{A}}$ to $\underline{\mathsf{B}}$ tells us that there is a full subcategory of $\underline{\mathsf{B}}$ which is *equivalent* to $\underline{\mathsf{A}}$ (in the formal sense of equivalences of categories). Once a (co)reflection is established, it is often interesting to identify such a subcategory. In the case of $\underline{\mathsf{HL}}$ and $\underline{\mathsf{ST}}$ the question is answered below.

Proposition 1.7 (*Languages are deterministic Trees*)
The full subcategory of $\underline{\mathsf{ST}}$ consisting of those synchronisation trees which are deterministic, say $\underline{\mathsf{dST}}$, is equivalent to the category of Hoare languages.

2 Deterministic Transition Systems

Speaking informally behaviour/system and linear/branching are independent parameters, and we expect to be able to forget the branching structure of a transition system without necessarily losing all the internal structure of the system. This leads us to identify a class of models able to represent the internal structure of processes without keeping track of their branching, i.e., the points at which the choices are actually taken. A suitable model is given by *deterministic transition systems*.

Definition 2.1 (*Deterministic Transition Systems*)
A transition system T is deterministic if

$$(s,a,s'),\ (s,a,s'') \in Tran_T \quad \textit{implies} \quad s' = s''.$$

Let $\underline{\mathsf{dTS}}$ be the full subcategory of $\underline{\mathsf{TS}}$ consisting of those transition systems which are deterministic.

Consider the binary relation $\simeq$ on the state of a transition system T defined as the least equivalence which is *forward closed*, i.e.,

$$s \simeq s' \text{ and } (s,a,u),\ (s',a,u') \in Tran_T \quad \Rightarrow \quad u \simeq u',$$

and define $ts.dts(T) = \big(S/\!\simeq, [s^I_T]_\simeq, L_T, Tran_\simeq\big)$, where $S/\!\simeq$ are the equivalence classes of $\simeq$ and

$$\big([s]_\simeq, a, [s']_\simeq\big) \in Tran_\simeq \quad \Leftrightarrow \quad \exists(\bar{s},a,\bar{s}') \in Tran_T \quad \text{with } \bar{s} \simeq s \text{ and } \bar{s}' \simeq s'.$$

It is easy to see that the transition system $ts.dts(TS)$ is deterministic. Actually, this construction defines a functor which is left adjoint to the inclusion

$\underline{\mathsf{dTS}} \hookrightarrow \underline{\mathsf{TS}}$. In the following we briefly prove this fact. Since confusion is never possible, we shall not use different notations for different $\simeq$'s.

Given a transition system morphism $(\sigma, \lambda): T_0 \to T_1$, define $ts.dts\big((\sigma, \lambda)\big)$ to be $(\bar{\sigma}, \lambda)$, where $\bar{\sigma}: S_{T_0}/\!\simeq\, \to S_{T_1}/\!\simeq$ is such that $\bar{\sigma}\big([s]_{\simeq}\big) = [\sigma(s)]_{\simeq}$.

Proposition 2.2 (*$ts.dts$: $\underline{\mathsf{TS}} \to \underline{\mathsf{dTS}}$ is a functor*)
The pair $(\bar{\sigma}, \lambda): ts.dts(T_0) \to ts.dts(T_1)$ is a transition system morphism.

Proof. We show that $\bar{\sigma}$ is well-defined. For $(s, a, s'),\ (s, a, s'') \in Tran_{T_0}$, if $\lambda\!\uparrow\! a$, then $\sigma(s') = \sigma(s) = \sigma(s'')$; otherwise, $\big(\sigma(s), \lambda(a), \sigma(s')\big),\ \big(\sigma(s), \lambda(a), \sigma(s'')\big) \in Tran_{T_1}$. Therefore, in both cases, $\sigma(s') \simeq \sigma(s'')$. Now, since $(s, a, s') \in Tran_{T_0}$ implies $\big(\sigma(s), \lambda(a), \sigma(s')\big) \in Tran_{T_1}$ or $\sigma(s) = \sigma(s')$, it follows that $\sigma(\simeq) \subseteq \simeq$. It is now easy to show that $(\bar{\sigma}, \lambda)$ is a morphism. ✓

It follows easily from the previous proposition that $ts.dts$ is a functor.

Clearly, for a deterministic transition system, say DT, there are no pairs $(s, a, s'),\ (s, a, s'') \in Tran_{DT}$ with $s' \neq s''$. Thus, $\simeq$ is the identity, and we can choose a candidate for the counit by considering, for any deterministic transition system DT, the morphism $(\varepsilon, id): ts.dts(DT) \to DT$, where $\varepsilon([s]_{\simeq}) = s$.

Proposition 2.3 (*$(\varepsilon, id): ts.dts(DT) \to DT$ is couniversal*)
For any deterministic transition system DT, any transition system T, and any morphism $(\eta, \lambda): ts.dts(T) \to DT$, there exists a unique k in $\underline{\mathsf{TS}}$ such that $(\varepsilon, id) \circ ts.dts(k) = (\eta, \lambda)$.

$$\begin{tikzcd}
ts.dts(DT) \arrow[r, "{(\varepsilon,id)}"] & DT \\
ts.dts(T) \arrow[u, "ts.dts(k)"] \arrow[ur, "{(\eta,\lambda)}"'] &
\end{tikzcd}$$

Proof. The morphism k must be of the form (σ, λ), for some σ. We choose σ such that $\sigma(s) = \eta([s]_{\simeq})$. This clearly makes k be a transition system morphism. Moreover, the diagram commutes: $(\varepsilon, id) \circ ts.dts\big((\sigma, \lambda)\big) = (\varepsilon \circ \bar{\sigma}, \lambda)$, and $\varepsilon\big(\bar{\sigma}([s]_{\simeq})\big) = \varepsilon([\sigma(s)]_{\simeq}) = \sigma(s) = \eta([s]_{\simeq})$. To show uniqueness of k, suppose that there is k' which makes the diagram commute. Necessarily, k' must be of the kind (σ', λ). Now, since $\sigma'([s]_{\simeq}) = [\sigma'(s)]_{\simeq}$, in order for the diagram to commute, it must be $\sigma'(s) = \eta([s]_{\simeq})$. Therefore, $\sigma' = \sigma$ and then $k' = k$. ✓

Theorem 2.4 (*$ts.dts \dashv \hookrightarrow$*)
The functor $ts.dts$ is left adjoint to the inclusion functor $\underline{\mathsf{dTS}} \hookrightarrow \underline{\mathsf{TS}}$. Therefore, the adjunction is a reflection.

Proof. By standard results of Category Theory (see [7, chap. IV, pg. 81]). ✓

Remark. It is worth noticing that $ts.dts$ does not coincide with the classical 'subset construction' of automata theory, which is not even functorial on $\underline{\mathsf{TS}}$. Our construction, as implied by the kind of simulations the morphisms of $\underline{\mathsf{TS}}$ are, preserves behaviours 'weakly': $ts.dts(T)$ simulates T, i.e., the behaviours of T are behaviours of $ts.dts(T)$, but not necessarily the converse, i.e., $ts.dts(T)$ may exhibit more behaviours (see, e.g., Example 5.1).

Next, we present a universal construction from Hoare languages to deterministic transition system, namely a coreflection $\underline{\mathsf{HL}} \hookrightarrow \underline{\mathsf{dTS}}$. Let (H, L) be a language. Define $hl.dts(H, L) = (H, \epsilon, L, \mathit{Tran})$, where $(s, a, sa) \in \mathit{Tran}$ for any $sa \in H$, which is trivially a deterministic transition system.

On the contrary, given a deterministic transition system DT, define the language $dts.hl(DT) = (\mathit{Act}(DT), L_{DT})$. Concerning morphisms, it is immediate that if $(\sigma, \lambda)\colon DT_0 \to DT_1$ is a transition system morphism, then $\lambda\colon \mathit{Act}(DT_0) \to \mathit{Act}(DT_1)$ is a morphism of Hoare languages. Of course then, defining $dts.hl\big((\sigma, \lambda)\big) = \lambda$, we have a functor from $\underline{\mathsf{dTS}}$ to $\underline{\mathsf{HL}}$.

Now, consider the language $dts.hl \circ hl.dts(H, L)$. It contains a string $a_1 \cdots a_n$ if and only if the sequence $(\epsilon, a_1, a_1)(a_1, a_2, a_1a_2) \cdots (a_1 \cdots a_{n-1}, a_n, a_1 \cdots a_n)$ is in $\mathit{Path}\big(hl.dts(T)\big)$ if and only if $a_1 \cdots a_n$ is in H. It follows immediately that $id\colon (H, L) \to dts.hl \circ hl.dts(H, L)$ is a morphism of languages. We will show that id is actually the unit of the coreflection.

Proposition 2.5 ($id\colon (H, L) \to dts.hl \circ hl.dts(H, L)$ *is universal*)
For any Hoare language (H, L), any deterministic transition system DT, and any morphism $\lambda\colon (H, L) \to dts.hl(DT)$, there exists a unique k in $\underline{\mathsf{dTS}}$ such that $dts.hl(k) = \lambda$.

$$
\begin{tikzcd}
(H, L) \arrow[r, "id"] \arrow[dr, "\lambda"'] & dts.hl \circ hl.dts(H, L) \arrow[d, "dts.hl(k)"] \\
& dts.hl(DT)
\end{tikzcd}
$$

Proof. Observe that since DT is deterministic, given a string $s \in \mathit{Act}(DT)$, there is exactly one state in S_{DT} reachable from s^I_{DT} with a path labelled by s. We shall use $\mathit{state}(s)$ to denote such a state. Then, define $k = (\sigma, \lambda)\colon hl.dts(H, L) \to DT$, where $\sigma(s) = \mathit{state}(\hat{\lambda}(s))$. Since DT is deterministic and $\hat{\lambda}(s)$ is in $\mathit{Act}(DT)$, (σ, λ) is well-defined and the rest of the proof follows easily. ✓

Theorem 2.6 ($hl.dts \dashv dts.hl$)
The map $hl.dts$ extends to a functor from $\underline{\mathsf{HL}}$ to $\underline{\mathsf{dTS}}$ which is left adjoint to $dts.hl$. Since the unit of the adjunction is an isomorphism, the adjunction is a coreflection.

Observe that the construction of the deterministic transition system associated to a language coincides exactly with the construction of the corresponding synchronisation tree. However, due to the different objects in the categories, the type of universality of the construction changes. In other words, the same construction shows that $\underline{\mathsf{HL}}$ is *reflective* in $\underline{\mathsf{ST}}$—a full subcategory of $\underline{\mathsf{TS}}$—and *coreflective* in $\underline{\mathsf{dTS}}$—another full subcategory of $\underline{\mathsf{TS}}$.

Thus, we enriched the diagram at the end of the previous section and we have a square.

Theorem 2.7 (*The Interleaving Surface*)

$$
\begin{tikzcd}
\underline{\mathsf{dTS}} \arrow[r, hook, "\lhd"'] & \underline{\mathsf{TS}} \\
\underline{\mathsf{HL}} \arrow[u, hook] \arrow[r, hook] & \underline{\mathsf{ST}} \arrow[u, hook]
\end{tikzcd}
$$

3 Non-Interleaving vs. Interleaving Models

Event structures [10, 21] abstract away from the cyclic structure of the process and consider only events (strictly speaking event *occurrences*), assumed to be the *atomic computational steps*, and the cause/effect relationships between them. Thus, we can classify event structures as *behavioural*, *branching* and *non-interleaving* models. Here, we are interested in labelled event structures.

Definition 3.1 (*Labelled Event Structures*)
A labelled event structure *is a structure $ES = (E, \#, \leq, \ell, L)$ consisting of a set of* events *E partially ordered by $\leq$; a symmetric, irreflexive relation $\# \subseteq E \times E$, the* conflict *relation, such that*

$$\{\, e' \in E \mid e' \leq e \,\} \text{ is finite for each } e \in E;$$

$$e \# e' \leq e'' \text{ implies } e \# e'' \text{ for each } e, e', e'' \in E;$$

a set of labels *L and a* labelling function *$\ell \colon E \to L$. For an event $e \in E$, define $\lfloor e \rfloor = \{\, e' \in E \mid e' \leq e \,\}$. Moreover, we write $\mathbin{\text{W}}$ for $\# \cup \{\, (e,e) \mid e \in E\}$. These data define a relation of* concurrency *on events: $co = E \times E \setminus (\leq \cup \leq^{-1} \cup \#)$.*
A labelled event structure morphism *from ES_0 to ES_1 is a pair of partial functions (η, λ), where $\eta \colon E_{ES_0} \rightharpoonup E_{ES_1}$ and $\lambda \colon L_{ES_0} \rightharpoonup L_{ES_1}$ are such that*

i) $\lfloor \eta(e) \rfloor \subseteq \eta(\lfloor e \rfloor)$, *if* $\eta{\downarrow}e$;

ii) $\eta(e) \mathbin{\text{W}} \eta(e')$ *implies* $e \mathbin{\text{W}} e'$, *if* $\eta{\downarrow}e$, $\eta{\downarrow}e'$;

iii) $\lambda \circ \ell_{ES_0} = \ell_{ES_1} \circ \eta$, *i.e., the following diagram commutes.*

$$
\begin{tikzcd}
E_{ES_0} \arrow[r, "\ell_{ES_0}"] \arrow[d, "\eta"'] & L_{ES_0} \arrow[d, "\lambda"] \\
E_{ES_1} \arrow[r, "\ell_{ES_1}"'] & L_{ES_1}
\end{tikzcd}
$$

This defines the category $\underline{\mathsf{LES}}$ *of labelled event structures.*

The computational intuition behind event structures is simple: an event e is *enabled* and can occur when all its *causes*, viz. $\lfloor e \rfloor \setminus \{e\}$, have occurred and no event which it is in conflict with has already occurred. This is formalised by the following notions of *configuration* and *enabling*. Notice that conditions *(i)* and *(ii)* above ensure precisely that morphisms of event structures preserve the computationally relevant structure, namely configurations and enabling.

DEFINITION 3.2 (*Configurations*)
Given a labelled event structure ES, define the configurations of ES to be those subsets $c \subseteq E_{ES}$ which are

$$\begin{aligned} \textit{Conflict Free:} \quad & \forall e_1, e_2 \in c, \ \textit{not } e_1 \# e_2; \\ \textit{Left Closed:} \quad & \forall e \in c \ \forall e' \leq e, \ e' \in c. \end{aligned}$$

Let $\mathcal{L}(ES)$ denote the set of configurations of ES.
We say that e is enabled at a configuration c, in symbols $c \vdash e$, if

$$\textit{(i)} \quad e \notin c; \quad \textit{(ii)} \quad \lfloor e \rfloor \setminus \{e\} \subseteq c; \quad \textit{(iii)} \quad e' \in E_{ES} \textit{ and } e' \# e \textit{ implies } e' \notin c.$$

The occurrence of e at c transforms c in the configuration $c' = c \cup \{e\}$.

Given a finite subset c of E_{ES}, we say that a total ordering of the elements of c, say $\{e_1 < e_2 < \cdots < e_n\}$, is a *securing* for c if and only if $\{e_1, \ldots, e_{i-1}\} \vdash e_i$, for $i = 1, \ldots, n$. Clearly, c is a finite configuration if and only if there exists a securing for it. We shall write a securing for c as a string $e_1 e_2 \cdots e_n$, where $c = \{e_1, e_2, \ldots, e_n\}$ and $e_i \neq e_j$ for $i \neq j$, and, by abuse of notation, we shall consider such strings also configurations. Let $Sec(ES)$ denote the set of the securings of ES.

DEFINITION 3.3 (*Deterministic Event Structures*)
A labelled event structure ES is deterministic if and only if for any $c \in \mathcal{L}(ES)$, and for any pair of events $e, e' \in E_{ES}$, whenever $c \vdash e$, $c \vdash e'$ and $\ell(e) = \ell(e')$, then $e = e'$.
This defines the category $\underline{\mathsf{dLES}}$ as a full subcategory of $\underline{\mathsf{LES}}$.

In [19], it is shown that synchronisation trees and labelled event structures are related by a coreflection from $\underline{\mathsf{ST}}$ to $\underline{\mathsf{LES}}$. As will be clear later, this gives us a way to see synchronisation trees as an interleaving version of labelled event structures or, vice versa, to consider labelled event structures as a generalisation of synchronisation trees to the non-interleaving case. In the following subsection, we give a brief account of this coreflection.

Synchronisation Trees and Labelled Event Structures

Given a tree S, define $st.les(S) = (Tran_S, \leq, \#, \ell, L_S)$, where

- $\leq$ is the least partial order on $Tran_S$ such that $(s, a, s') \leq (s', b, s'')$;
- $\#$ is the least hereditary, symmetric, irreflexive relation on $Tran_S$ such that $(s, a, s') \# (s, b, s'')$ if $s' \neq s''$;

- $\ell\big((s,a,s')\big) = a$.

It is clear that $st.les(S)$ is a labelled event structure. Now, by defining $st.les\big((\sigma,\lambda)\big) = (\eta_\sigma, \lambda)$, where

$$\eta_\sigma\big((s,a,s')\big) = \begin{cases} \big(\sigma(s), \lambda(a), \sigma(s')\big) & \text{if } \lambda{\downarrow}a \\ \uparrow & \text{otherwise,} \end{cases}$$

we extend $st.les$ to a functor from $\underline{\mathsf{ST}}$ to $\underline{\mathsf{LES}}$.

On the contrary, for a labelled event structure ES, define $les.st(ES)$ to be the structure $\big(Sec(ES), \epsilon, L_{ES}, Tran\big)$, where $(s,a,se) \in Tran$ if and only if $s,\ se \in Sec(ES)$ and $\ell_{ES}(e) = a$. Since the existence of a transition (s,a,s') implies that s is a string strictly shorter than s', the transition system we obtain is certainly acyclic. Moreover, by definition of securing, it is reachable. Finally, if $(s,a,se),\ (s',a,s'e') \in Tran$ and $se = s'e'$, then obviously $s = s'$ and $e = e'$. Therefore, $les.st(ES)$ is a synchronisation tree.

Concerning morphisms, for $(\eta,\lambda)\colon ES_0 \to ES_1$, define $les.st\big((\eta,\lambda)\big)$ to be $(\hat{\eta},\lambda)$. This makes $les.st$ be a functor from $\underline{\mathsf{LES}}$ to $\underline{\mathsf{ST}}$.

Consider now $les.st \circ st.les(S)$. Observe that there is a transition

$$\Big((s_S^I, a_1, s_1)\cdots(s_{n-1}, a_n, s_n),\ a,\ (s_S^I, a_1, s_1)\cdots(s_{n-1}, a_n, s_n)(s_n, a, s)\Big)$$

in $Tran_{les.st \circ st.les(S)}$ if and only if $(s_S^I, a_1, s_1)\cdots(s_{n-1}, a_n, s_n)(s_n, a, s)$ is a path in S. From this, and since S and $les.st \circ st.les(S)$ are trees, it follows that there is an isomorphism between the states of S and the states of $les.st \circ st.les(S)$, and that such an isomorphism is indeed a morphism of synchronisation trees.

Theorem 3.4 ($st.les \dashv les.st$)
For any synchronisation tree S, the map $(\eta, id)\colon S \to les.st \circ st.les(S)$, where $\eta(s_S^I) = \epsilon$ and $\eta(s) = (s_S^I, a_1, s_1)\cdots(s_n, a, s)$, the unique path leading to s in S, is a synchronisation tree isomorphism.

Moreover, $\langle st.les, les.st\rangle\colon \underline{\mathsf{ST}} \rightharpoonup \underline{\mathsf{LES}}$ is an adjunction whose unit is given by the family of isomorphisms (η, id). Thus, we have a coreflection of $\underline{\mathsf{ST}}$ into $\underline{\mathsf{LES}}$.

Consider now a synchronisation tree S in $\underline{\mathsf{dST}}$, i.e., a deterministic tree. From the definition of $st.les$, it follows easily that $st.les(S)$ is a deterministic event structure; on the other hand, $les.st(ES)$ is a deterministic tree when ES is deterministic. Thus, by general reason, the coreflection $\underline{\mathsf{ST}} \hookrightarrow \underline{\mathsf{LES}}$ restricts to a coreflection $\underline{\mathsf{dST}} \hookrightarrow \underline{\mathsf{dLES}}$, whence we have the following corollary.

Theorem 3.5 ($\underline{\mathsf{HL}} \hookrightarrow \underline{\mathsf{dLES}}$)
The category $\underline{\mathsf{HL}}$ of Hoare languages is coreflective in the category $\underline{\mathsf{dLES}}$ of deterministic labelled event structures.

Proof. It is enough to recall that $\underline{\mathsf{dST}}$ and $\underline{\mathsf{HL}}$ are *equivalent.* ✓

To conclude this subsection, we make precise our claim of labelled event structures being a generalisation of synchronisation trees to the non-interleaving case. Once the counits of the above coreflections have been calculated, it is not difficult to prove the following results.

Corollary 3.6 (*L. Event Structures = S. Trees + Concurrency*)
The full subcategory of $\underline{\mathsf{LES}}$ *consisting of the labelled event structures ES such that* $co_{ES} = \varnothing$ *is equivalent to* $\underline{\mathsf{ST}}$.

The full subcategory of $\underline{\mathsf{dLES}}$ *consisting of the deterministic labelled event structures ES such that* $co_{ES} = \varnothing$ *is equivalent to* $\underline{\mathsf{HL}}$.

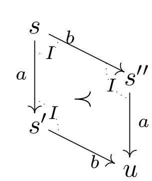
Transition Systems with Independence

Now, on the system level we look for a way of equipping transition systems with a notion of 'concurrency' or 'independence', in the same way as $\underline{\mathsf{LES}}$ may be seen as adding 'concurrency' to $\underline{\mathsf{ST}}$. Moreover, such enriched transition systems should also represent the 'system model' version of event structures. Several such models have appeared in the literature [17, 1, 18, 3]. However, the asynchronous automata of [17] are not suited to our programme, since they are inherently *deterministic*. Also the transition systems introduced in [1, 18, 3] do not fit directly the frame, as they are *unlabelled*. Nevertheless, we could use them profitably provided a layer of labels is added on top of the events which decorate their transitions. However, since such a double 'decoration' of transitions would not be mathematically very pleasant, here we choose a variation of these notions, the *transition systems with independence* [22].

Transition systems with independence are labelled transition systems with an independence relation carried by the transitions. The novelty resides in the fact that the notion of *event* becomes a derived notion. However, four axioms are imposed in order to guarantee the consistency of this with the intuitive meaning of event.

Definition 3.7 (*Transition Systems with Independence*)
A transition system with independence is a structure $(S, s^I, L, Tran, I)$, *where* $(S, s^I, L, Tran)$ *is a transition system and* $I \subseteq Tran^2$ *is an irreflexive, symmetric relation, such that, using* $\prec$ *to denote the following relation on transitions*

$$\begin{aligned}(s,a,s') \prec (s'',a,u) \quad \Leftrightarrow \quad \exists b.\ & (s,a,s')\ I\ (s,b,s'') \text{ and} \\ & (s,a,s')\ I\ (s',b,u) \text{ and} \\ & (s,b,s'')\ I\ (s'',a,u),\end{aligned}$$



and $\sim$ *for least equivalence containing* $\prec$, *we have*

i) $(s,a,s') \sim (s,a,s'') \quad \Rightarrow \quad s' = s''$;

ii) $(s,a,s')\ I\ (s,b,s'') \quad\Rightarrow\quad \exists u.\ (s,a,s')\ I\ (s',b,u)$ and $(s,b,s'')\ I\ (s'',a,u)$;

i.e.,

```latex
\begin{tikzcd}[column sep=small]
 & s \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "I" near start] & \\
s' & \phantom{x} & s''
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}[column sep=small]
 & s \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "I" near start] & \\
s' \arrow[dr, "b"'] \arrow[r, phantom, "I" near start] & \phantom{x} & s'' \arrow[dl, "a"] \arrow[l, phantom, "I" near start] \\
 & u &
\end{tikzcd}
```

iii) $(s,a,s')\ I\ (s',b,u) \quad\Rightarrow\quad \exists s''.\ (s,a,s')\ I\ (s,b,s'')$ and $(s,b,s'')\ I\ (s'',a,u)$;

i.e.,

```latex
\begin{tikzcd}[column sep=small]
 & s \arrow[dl, "a"'] & \\
s' \arrow[dr, "b"'] \arrow[r, phantom, "I" near start] & \phantom{x} & \\
 & u &
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}[column sep=small]
 & s \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "I" near start] & \\
s' \arrow[dr, "b"'] \arrow[r, phantom, "I" near start] & \phantom{x} & s'' \arrow[dl, "a"] \arrow[l, phantom, "I" near start] \\
 & u &
\end{tikzcd}
```

iv) $(s,a,s') \sim (s'',a,u)\ I\ (w,b,w') \quad\Rightarrow\quad (s,a,s')\ I\ (w,b,w')$;

Morphisms of transition systems with independence are morphisms of the underlying transition systems which preserve independence, i.e., such that

$$(s,a,s')\ I\ (\bar{s},b,\bar{s}') \text{ and } \lambda{\downarrow}a,\ \lambda{\downarrow}b \quad\Rightarrow\quad \big(\sigma(s),\lambda(a),\sigma(s')\big)\ I\ \big(\sigma(\bar{s}),\lambda(b),\sigma(\bar{s}')\big).$$

These data define the category $\underline{\mathsf{TSI}}$ *of transition systems with independence. Moreover, let* $\underline{\mathsf{dTSI}}$ *denote the full subcategory of* $\underline{\mathsf{TSI}}$ *consisting of transition systems with independence whose underlying transition system is deterministic.*

Thus, transition systems with independence are precisely standard transition systems but with an additional relation expressing when one transition is independent of another. The relation $\sim$, defined as the reflexive, symmetric and transitive closure of a relation $\prec$ which simply identifies local 'diamonds' of concurrency, expresses when two transitions represent occurrences of the same event. Thus, the equivalence classes $[(s,a,s')]_\sim$ of transitions (s,a,s') are the *events* of the transition system with independence. In order to shorten notations, we shall indicate that transitions (s,a,s'), (s,b,s''), (s',b,u) and (s'',a,u) form a diamond by writing $Diam_{a,b}(s,s',s'',u)$.

Concerning the axioms, property *(i)* states that the occurrence of an event at a state yields a unique state; property *(iv)* asserts that the independence relation respects events. Finally, conditions *(ii)* and *(iii)* describe intuitive properties of independence: two independent events which can occur at the same state, can do it in any order without affecting the reached state.

Transition systems with independence admit $\underline{\mathsf{TS}}$ as a coreflective subcategory. In this case, the adjunction is easy. The left adjoint associates to any transition system T the transition system with independence whose underlying transition system is T itself and whose independence relation is *empty*. The right adjoint simply forgets about the independence, mapping any transition system with independence to its *underlying* transition system. From the definition of morphisms of transition systems with independence, it follows easily that these mappings extend to functors which form a coreflection $\underline{\mathsf{TS}} \hookrightarrow \underline{\mathsf{TSI}}$. Moreover, such a coreflection trivially restricts to a coreflection $\underline{\mathsf{dTS}} \hookrightarrow \underline{\mathsf{dTSI}}$.

So, we are led to the following diagram.

THEOREM 3.8 (*Moving along the 'interleaving/noninterleaving' axis*)

$$
\begin{tikzcd}
 & \underline{\mathsf{TSI}} & & \underline{\mathsf{TS}} \arrow[ll, hook] \\
\underline{\mathsf{dTSI}} & & \underline{\mathsf{dTS}} \arrow[ll, hook] \arrow[ur, hook] & \\
 & \underline{\mathsf{LES}} & & \underline{\mathsf{ST}} \arrow[ll, hook] \arrow[uu, hook] \\
\underline{\mathsf{dLES}} & & \underline{\mathsf{HL}} \arrow[ll, hook] \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

4 Transition Systems with Independence and Labelled Event Structures

In this section, we show that transition systems with independence are an extension of labelled event structures to a system model, by showing that there exists a coreflection from $\underline{\mathsf{LES}}$ to $\underline{\mathsf{TSI}}$. To simplify our task, we split such a coreflection in two parts. First, we define the *unfolding* of transition systems with independence. To this aim, we introduce the category $\underline{\mathsf{oTSI}}$ of occurrence transition systems with independence, obtained from $\underline{\mathsf{TSI}}$ via conditions reminiscent of those which yield trees from transition systems. Later, we shall show that labelled event structures are coreflective in $\underline{\mathsf{oTSI}}$, thus obtaining

$$
\underline{\mathsf{LES}} \hookrightarrow \underline{\mathsf{oTSI}} \hookrightarrow \underline{\mathsf{TSI}}.
$$

In addition, we shall identify a subcategory of $\underline{\mathsf{oTSI}}$ equivalent to $\underline{\mathsf{LES}}$, so yielding an account of coherent, finitary, prime algebraic domains in terms of transition systems.

DEFINITION 4.1 (*Occurrence Transition Systems with Independence*)
An occurrence transition system with independence is a transition system with independence $OTI = (S, s^I, L, Tran, I)$ which is reachable, acyclic and *such that*

$$
\begin{aligned}
(s', a, u) \neq (s'', b, u) \in Tran \quad & \textit{implies} \\
& \exists s.\ (s, b, s')\ I\ (s, a, s'') \textit{ and } (s, b, s')\ I\ (s', a, u) \\
& \qquad \textit{and } (s, a, s'')\ I\ (s'', b, u),
\end{aligned}
$$

i.e., $\begin{tikzcd} s' \arrow[dr, "b"'] & & s'' \arrow[dl, "a"] \\ & u & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & s \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "I" near start] & \\ s' \arrow[dr, "b"'] \arrow[r, phantom, "I" near start] & {} & s'' \arrow[dl, "a"] \arrow[l, phantom, "I" near start] \\ & u & \end{tikzcd}$

or, in other words, (s', a, u) and (s'', b, u) form the bottom of a concurrency diamond $Diam_{a,b}(s, s'', s', u)$.
Let <u>oTSI</u> *denote the full subcategory of* <u>TSI</u> *whose objects are occurrence transition systems with independence.*

Given a transition system with independence TI, define $\simeq \subseteq Path(TI)^2$ to be the least equivalence relation such that

$$\pi_s(s, a, s')(s', b, u)\pi_v \simeq \pi_s(s, b, s'')(s'', a, u)\pi_v \quad \text{if} \quad Diam_{a,b}(s, s', s'', u).$$

The following are some key, easy to prove, properties of occurrence transition systems with independence.

LEMMA 4.2
Given an occurrence transition system with independence OTI, let u be a state and π_u, π'_u paths leading to it. Then $\pi_u \simeq \pi'_u$.

Proof. By induction on the minimum length among those of π_u and π'_u.

If $|\pi_u| = |\pi'_u| = 0$, then $\pi_u = \epsilon = \pi'_u$.

Suppose that $\pi_u = \pi_{s'}(s', a, u)$, $\pi'_u = \pi_{s''}(s'', b, u)$ and suppose that $|\pi_{s'}| \leq |\pi_{s''}|$. Then, necessarily, it must be $Diam_{a,b}(s, s'', s', u)$, for some $s \in S_{OTI}$. Since OTI is reachable, there exists a path $\pi_0 = \pi_s(s, b, s')$. Since the length of $\pi_{s'}$ is $n-1$, we have that $\min\{|\pi_0|, |\pi_{s'}|\} \leq n-1$. So, we can apply the induction hypothesis and conclude that $\pi_{s'} \simeq \pi_0$. From the definition of $\simeq$, it follows that π_0 has length $n-1$. Thus, $\pi_1 = \pi_s(s, a, s'')$ has length $n-1$ and, by induction, $\pi_1 \simeq \pi_{s''}$. So, $\pi_u \simeq \pi_s(s, b, s')(s', a, u) \simeq \pi_s(s, a, s'')(s'', b, u) \simeq \pi'_u$. ✓

COROLLARY 4.3
Any pair of sequences leading from state $\bar{s}$ to state $\bar{s}'$ of OTI contain the same number of representatives of any $\sim$-equivalence class.

Proof. First suppose that $\bar{s}$ is the initial state s^I_{OTI}. Then the sequences are two paths leading to the same state and therefore, by Lemma 4.2, they are $\simeq$-equivalent. In the case $\pi_s(s, a, s')(s', b, u)\pi_{\bar{s}'} \simeq \pi_s(s, b, s'')(s'', a, u)\pi_{\bar{s}'}$, the result is immediate, since $(s, a, s') \sim (s'', a, u)$ and $(s, b, s'') \sim (s', b, u)$. In the general case, the result follows by applying transitively the previous argument.

Now, consider two sequences from a generic $\bar{s}$ to $\bar{s}'$, say $\sigma_{\bar{s} \to \bar{s}'}$ and $\sigma'_{\bar{s} \to \bar{s}'}$. If there were a $\sim$-class whose elements occur a different number of times in $\sigma_{\bar{s} \to \bar{s}'}$ and $\sigma'_{\bar{s} \to \bar{s}'}$, then the same would happen for the paths $\pi_s \sigma_{\bar{s} \to \bar{s}'}$ and $\pi_s \sigma'_{\bar{s} \to \bar{s}'}$, and that would contradict what we have just shown in the first part of this proof. ✓

COROLLARY 4.4
If (s, a, s') and (s, b, s') are transitions of OTI, then $a = b$.

Proof. By reachability and by Lemma 4.2, we have $\pi_s(s, a, s') \simeq \pi_s(s, b, s')$. It follows then from Lemma 4.3 that $(s, a, s') \sim (s, b, s')$, and so $a = b$. ✓

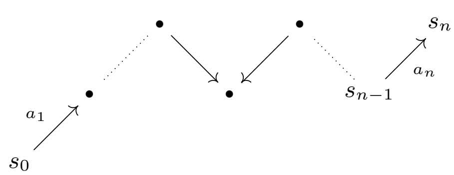
Summing up, occurrence transition systems with independence are very well structured and regular. In particular, the last result implies that in an occurrence transition system with independence each diamond of concurrency is *not* degenerate, i.e., it consists of four *distinct* states.

The next step is to show that in a path of an occurrence transition system with independence at most one representative of a $\sim$-class may appear. Given a path π and an equivalence class $[(s,a,s')]_\sim$, let $\mathcal{N}\big(\pi,[(s,a,s')]_\sim\big)$ be the number of representatives of $[(s,a,s')]_\sim$ occurring in π. Since we know from Corollary 4.3 that such a number depends on π only by means of the state it reaches, we shall write simply $\mathcal{N}\big(x,[(s,a,s')]_\sim\big)$, for $x \in S_{OTI}$. Moreover, let $s \overset{a}{\longleftrightarrow} s'$ stand for $s \xrightarrow{a} s'$ or $s \xleftarrow{a} s'$. Then we have the following result.

Lemma 4.5
Consider a sequence of states $\sigma = s_0 \overset{a_1}{\longleftrightarrow} s_1 \overset{a_2}{\longleftrightarrow} s_2 \cdots \overset{a_n}{\longleftrightarrow} s_n$. *Then*

$$\begin{aligned}\mathcal{N}\big(s_n,[(s,a,s')]_\sim\big) = \mathcal{N}\big(s_0,[(s,a,s')]_\sim\big)&\\ + \Big|\big\{(s_i,a_{i+1},s_{i+1}) \;\big|\; (s_i,a_{i+1},s_{i+1}) \sim (s,a,s')\big\}\Big|&\\ - \Big|\big\{(s_{i+1},a_{i+1},s_i) \;\big|\; (s_{i+1},a_{i+1},s_i) \sim (s,a,s')\big\}\Big|&.\end{aligned}$$

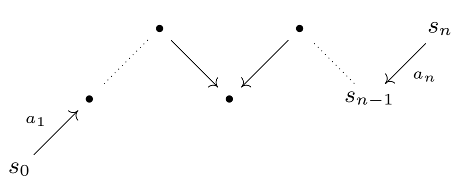
Proof. By induction on n, the length of σ. For $n = 0$, σ is empty and the thesis is trivially true. Suppose then that the thesis holds for sequences of length $n-1$. There are two cases: $s_{n-1} \xrightarrow{a_n} s_n$ or $s_n \xrightarrow{a_n} s_{n-1}$.



(case $s_{n-1} \xrightarrow{a_n} s_n$.) If $(s_{n-1},a_n,s_n) \not\sim (s,a,s')$ then $\mathcal{N}\big(s_n,[(s,a,s')]_\sim\big) = \mathcal{N}\big(s_{n-1},[(s,a,s')]_\sim\big)$, and since nothing is added to or subtracted from the right hand term, the equality holds. If otherwise $(s_{n-1},a_n,s_n) \sim (s,a,s')$, then

$$\mathcal{N}\big(s_n,[(s,a,s')]_\sim\big) = \mathcal{N}\big(s_{n-1},[(s,a,s')]_\sim\big) + 1,$$

and the equality stays since 1 is added also to the right hand term. So, the induction hypothesis is maintained.

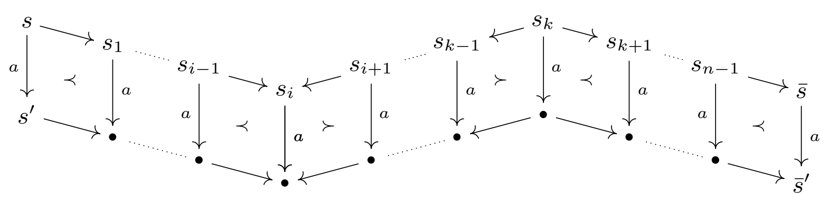


(case $s_n \xrightarrow{a_n} s_{n-1}$.) Again, if $(s_{n-1},a_n,s_n) \not\sim (s,a,s')$ the terms on both the sides of the equation are unchanged considering the n-th transition, and the result holds by induction. Otherwise if $(s_{n-1},a_n,s_n) \sim (s,a,s')$, then $\mathcal{N}\big(s_n,[(s,a,s')]_\sim\big) = \mathcal{N}\big(s_{n-1},[(s,a,s')]_\sim\big) - 1$. This time 1 is subtracted from the right hand term, and therefore the induction hypothesis is maintained. ✓

Then, we have the following important corollary.

Corollary 4.6
Given a path $\pi \in Path(OTI)$, at most one representative of any $\sim$-equivalence class can occur in π.

Proof. Suppose that $(s,a,s') \sim (\bar{s},a,\bar{s}')$ occur both in π. By definition of $\sim$, it must exist a sequence $\sigma = \big(s = s_0 \xleftrightarrow{a_0} \cdots \xleftrightarrow{a_n} s_n = \bar{s}\big)$, as shown by the following figure.



Without loss of generality, we can assume $\pi = \pi'(s,a,s')\sigma'(\bar{s},a,\bar{s}')\sigma''$, i.e., that (s,a,s') occurs before $(\bar{s},a,\bar{s}')$. Now, since (s,a,s') appears in π after state s, we have

$$\mathcal{N}\big(s, [(s,a,s')]_\sim\big) < \mathcal{N}\big(\bar{s}, [(s,a,s')]_\sim\big).$$

By the previous lemma, we have that in σ at least a representative of $[(s,a,s')]_\sim$ must occur *'positively'*, say $(s_k, a_{k+1}, s_{k+1}) \sim (s,a,s')$. Therefore, we have a diamond $Diam_{a_{k+1},a}(s_k, s_{k+1}, \bar{s}_k, \bar{s}_{k+1})$ where, from the property shown earlier, it must be $s_k \neq \bar{s}_k$. This is absurd, because $(s_k, a_{k+1}, s_{k+1}) \sim (s_k, a, \bar{s}_k)$ breaks axiom *(i)* of transition systems with independence. ✓

Unfolding Transition Systems with Independence

Given a transition system with independence $TI = (S, s^I, L, Tran, I)$, we define $tsi.otsi(TI) = \big(\Pi_\simeq, [\epsilon]_\simeq, L, Tran_\simeq, I_\simeq\big)$, where

- $\Pi_\simeq$ is the quotient of $Path(TI)$ modulo $\simeq$;
- $\big([\pi]_\simeq, a, [\pi']_\simeq\big) \in Tran_\simeq \quad\Leftrightarrow\quad \exists (s,a,s') \in Tran$ such that $\pi' \simeq \pi(s,a,s')$;
- $\big([\pi]_\simeq, a, [\pi']_\simeq\big)\ I_\simeq\ \big([\bar{\pi}]_\simeq, b, [\bar{\pi}']_\simeq\big) \quad\Leftrightarrow$

  $\exists (s,a,s')\ I\ (\bar{s},b,\bar{s}') \in Tran$ such that $\pi' \simeq \pi(s,a,s')$, and $\bar{\pi}' \simeq \bar{\pi}(\bar{s},b,\bar{s}')$.

Proposition 4.7
The transition system $tsi.otsi(TI)$ is an occurrence transition system with independence.

Proof. We show only the condition in Definition 4.1 of occurrence transition systems with independence. Suppose that $\big([\pi']_\simeq, b, [\pi]_\simeq\big) \neq \big([\pi'']_\simeq, a, [\pi]_\simeq\big)$. Then, we have $\pi \simeq \pi'(s',b,u) \simeq \pi''(s'',a,u)$ with $\pi' \neq \pi''$. By definition of $\simeq$, it must exist $\bar{\pi}$ such that $\pi'(s',b,u) \simeq \bar{\pi}(s,a,s')(s',b,u)$ and $\pi''(s'',a,u) \simeq \bar{\pi}(s,b,s'')(s'',a,u)$. Moreover, it must be $\bar{\pi}(s,a,s') \simeq \pi'$ and $\bar{\pi}(s,b,s'') \simeq \pi''$.

Then, $\big([\bar{\pi}]_\simeq, a, [\bar{\pi}(s,a,s')]_\simeq\big)$ and $\big([\bar{\pi}]_\simeq, b, [\bar{\pi}(s,b,s'')]_\simeq\big)$ close the diamond. ✓

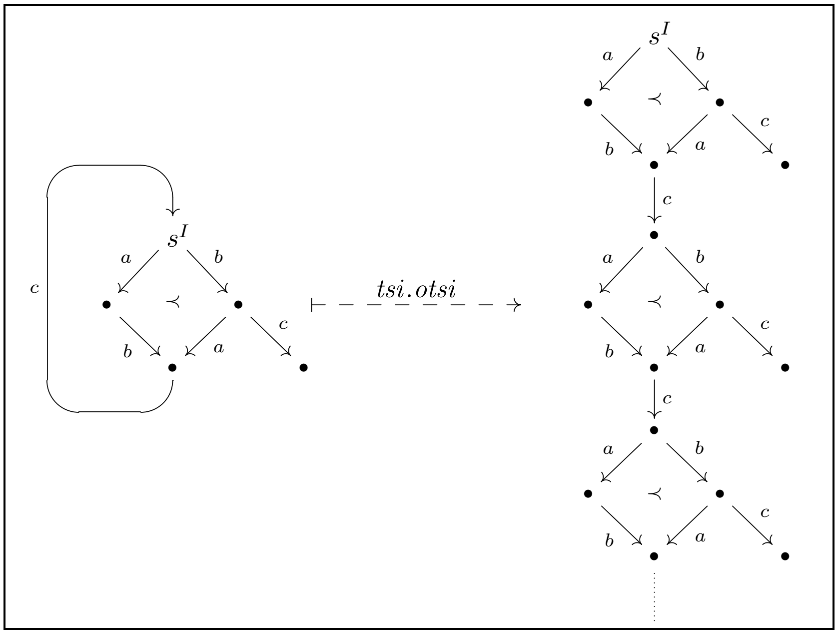


Figure 1: A transition system with independence TI and $tsi.otsi(TI)$.

Figure 1 shows a simple example of unfolding of a transition system with independence. Next, we show that $tsi.otsi$ extends to a functor for $\underline{\mathsf{TSI}}$ to $\underline{\mathsf{oTSI}}$ which is right adjoint to the inclusion functor $\underline{\mathsf{oTSI}} \hookrightarrow \underline{\mathsf{TSI}}$. As a candidate for the counit of such an adjunction, consider the mapping $(\sigma_\varepsilon, id)\colon tsi.otsi(TI) \to TI$, where

$$\sigma_\varepsilon(\epsilon) = s^I_{TI} \qquad \text{and} \qquad \sigma_\varepsilon\big([\pi_s]_\simeq\big) = s.$$

By definition of $\simeq$, we know that σ_ε is well-defined. Then, it is not difficult to see that (σ_ε, id) is a morphism of transition systems with independence.

Proposition 4.8 (*$(\sigma_\varepsilon, id)\colon tsi.otsi(TI) \to TI$ is couniversal*)
For any occurrence transition system with independence OTI, transition system with independence TI, and morphism $(\sigma, \lambda)\colon OTI \to TI$, there exists a unique $k\colon OTI \to tsi.otsi(TI)$ in $\underline{\mathsf{oTSI}}$ such that $(\sigma_\varepsilon, id) \circ k = (\sigma, \lambda)$.

$$\begin{tikzcd} tsi.otsi(TI) \arrow[r, "(\sigma_\varepsilon, id)"] & TI \\ OTI \arrow[u, "k"] \arrow[ur, "(\sigma,\lambda)"'] & \end{tikzcd}$$

Proof. Clearly, in order for the diagram to commute, k must be of the form $(\bar{\sigma}, \lambda)$. Consider the map $\bar{\sigma}(s) = [\sigma_\lambda(\pi_s)]_\simeq$, where $\sigma_\lambda\colon Path(OTI) \to Path(TI)$ is given by

$$\sigma_\lambda(\epsilon) = \epsilon; \qquad \sigma_\lambda\big(\pi_s(s,a,s')\big) = \begin{cases} \sigma_\lambda(\pi_s)\big(\sigma(s), \lambda(a), \sigma(s')\big) & \text{if } \lambda{\downarrow}a \\ \sigma_\lambda(\pi_s) & \text{otherwise.} \end{cases}$$

This definition is well-given: fixed s, let π_s and π'_s be two paths leading to s. Then, since OTI is an occurrence transition system with independence, it is $\pi_s \simeq \pi'_s$, and since (σ,λ) is a morphism, it is $\sigma_\lambda(\pi_s) \simeq \sigma_\lambda(\pi'_s)$. In order to show this last statement, it is enough to prove that

$$\begin{aligned} &\pi_s(s,a,s')(s',b,u)\pi_v \simeq \pi_s(s,b,s'')(s'',a,u)\pi_v \\ &\qquad \Rightarrow \quad \sigma_\lambda(\pi_s)\sigma_\lambda\big((s,a,s')(s',b,u)\big)\sigma_\lambda(\pi_v) \\ &\qquad\qquad\qquad \simeq \quad \sigma_\lambda(\pi_s)\sigma_\lambda\big((s,b,s'')(s'',a,u)\big)\sigma_\lambda(\pi_v). \end{aligned}$$

There are four cases:

i) $\lambda{\uparrow}a$, $\lambda{\uparrow}b$: then $\sigma_\lambda\big((s,a,s')(s',b,u)\big) = \epsilon = \sigma_\lambda\big((s,b,s'')(s'',a,u)\big)$, and the thesis follows easily.

ii) $\lambda{\downarrow}a$, $\lambda{\uparrow}b$: then

$$\begin{aligned} \sigma_\lambda\big((s,a,s')(s',b,u)\big) &= \big(\sigma(s), \lambda(a), \sigma(s')\big) \\ &= \big(\sigma(s''), \lambda(a), \sigma(u)\big) = \sigma_\lambda\big((s,b,s'')(s'',a,u)\big) \end{aligned}$$

and again the thesis follows.

iii) $\lambda{\uparrow}a$, $\lambda{\downarrow}b$: as for the previous point.

iv) $\lambda{\downarrow}a$, $\lambda{\downarrow}b$: then the thesis follows directly from the definition of morphism, since it is $Diam_{a,b}(s,s',s'',u)$ and in this case diamonds are preserved.

Let us show that $(\bar{\sigma},\lambda)$ is indeed a morphism of occurrence transition systems with independence.

i) $\bar{\sigma}(s^I_{OTI}) = [\epsilon]_\simeq$.

ii) Let $(s,a,s') \in Tran_{OTI}$, and suppose $\lambda{\downarrow}a$. Since OTI is reachable, we have $\pi_s(s,a,s') \in Path(OTI)$, and $\sigma_\lambda(\pi_s)\big(\sigma(s),\lambda(a),\sigma(s')\big)$ in $Path(TI)$. Thus, $\big([\sigma_\lambda(\pi_s)]_\simeq, \lambda(a), [\sigma_\lambda(\pi_s(s,a,s'))]_\simeq\big) = \big(\bar{\sigma}(s),\lambda(a),\bar{\sigma}(s')\big) \in Tran_\simeq$.

iii) If $(s,a,s')\ I_{OTI}\ (\bar{s},b,\bar{s}')$, $\big(\sigma(s),\lambda(a),\sigma(s')\big)\ I_{TI}\ \big(\sigma(\bar{s}),\lambda(b),\sigma(\bar{s}')\big)$, and reasoning as before, we get $\big(\bar{\sigma}(s),\lambda(a),\bar{\sigma}(s')\big)\ I_\simeq\ \big(\bar{\sigma}(\bar{s}),\lambda(b),\bar{\sigma}(\bar{s}')\big)$.

In order to show that the diagram commutes, it is enough to observe that each s is mapped to a $\simeq$-class of paths leading to $\sigma(s)$. Therefore, $\sigma_\varepsilon \circ \bar{\sigma}(s) = \sigma(s)$. The uniqueness of $(\bar{\sigma},\lambda)$ is easily obtained following the same argument. In fact, the behaviour of $\bar{\sigma}$ is compelled on any s: s^I_{OTI} must be mapped to $[\epsilon]_\simeq$, while a generic s must mapped to a $\simeq$-equivalence class of paths leading to $\sigma(s)$. But we know that there is a unique such class. ✓

THEOREM 4.9 ($\hookrightarrow \dashv tsi.otsi$)
The construction $tsi.otsi$ extends to a functor from $\underline{\mathsf{TSI}}$ to $\underline{\mathsf{oTSI}}$ which is right adjoint to the inclusion $\underline{\mathsf{oTSI}} \hookrightarrow \underline{\mathsf{TSI}}$.

It will be useful later to notice that this coreflection cuts down to a coreflections $\underline{\mathsf{doTSI}} \hookrightarrow \underline{\mathsf{dTSI}}$, where $\underline{\mathsf{doTSI}}$ is the full subcategory of $\underline{\mathsf{oTSI}}$ consisting of deterministic transition systems. In order to achieve this result, it is clearly enough to show that $tsi.otsi$ maps objects from $\underline{\mathsf{dTSI}}$ to $\underline{\mathsf{doTSI}}$.

Proposition 4.10 ($\underline{\mathsf{doTSI}} \hookrightarrow \underline{\mathsf{dTSI}}$)
If TI is deterministic, then $tsi.otsi(TI)$ is deterministic.

Proof. Suppose that $\big([\pi]_\simeq, a, [\pi']_\simeq\big)$ and $\big([\pi]_\simeq, a, [\pi'']_\simeq\big)$ are in $Tran_\simeq$. Then, it must be $\pi' \simeq \pi_s(s, a, s')$ and $\pi'' \simeq \pi_s(s, a, s'')$, for $(s, a, s'),\ (s, a, s'') \in Tran$. Then we have $s' = s''$ and so $\pi' \simeq \pi''$. ✓

Occurrence TSI's and Labelled Event Structures

In this subsection we complete the construction of the coreflections $\underline{\mathsf{LES}} \hookrightarrow \underline{\mathsf{TSI}}$ and $\underline{\mathsf{dLES}} \hookrightarrow \underline{\mathsf{dTSI}}$ by showing the existence of coreflections $\underline{\mathsf{LES}} \hookrightarrow \underline{\mathsf{oTSI}}$ and $\underline{\mathsf{dLES}} \hookrightarrow \underline{\mathsf{doTSI}}$, reminiscent of the connection between event structures and domains of configurations [10, 21].

Consider a labelled event structure $ES = (E, \leq, \#, \ell, L)$. Define $les.otsi(ES)$ to be the transition system with independence of the *finite configurations* of ES, i.e.,

$$les.otsi(ES) = \big(\mathcal{L}_F(ES), \varnothing, L, Tran, I\big),$$

where

- $\mathcal{L}_F(ES)$ is the set of finite configuration of ES;
- $(c, a, c') \in Tran$ if and only if $c = c' \smallsetminus \{e\}$ and $\ell(e) = a$;
- $(c, a, c')\ I\ (\bar{c}, b, \bar{c}')$ if and only if $(c' \smallsetminus c)\ co\ (\bar{c}' \smallsetminus \bar{c})$.

By definition, $les.otsi(ES)$ is clearly an acyclic, reachable transition system. Moreover, $I \subseteq Tran^2$ is symmetric and irreflexive, since co is such. In order to show that it is an occurrence transition system with independence, it is important the following characterisation of the relation $\sim$.

Lemma 4.11
Given (c, a, c') and $(\bar{c}, a, \bar{c}') \in Tran$, we have $(c, a, c') \sim (\bar{c}, a, \bar{c}') \in Tran$ if and only if $(c' \smallsetminus c) = (\bar{c}' \smallsetminus \bar{c})$.

Proof. ($\Rightarrow$). It is enough to show that $Diam_{a,b}(c, c', \bar{c}, \bar{c}')$ implies $(c' \smallsetminus c) = (\bar{c}' \smallsetminus \bar{c})$. Since $(c, a, c')\ I\ (c, b, \bar{c})$, we have $\{e\} = (c' \smallsetminus c)\ co\ (\bar{c} \smallsetminus c) = \{e'\}$. Let e'' be the event in $\bar{c}' \smallsetminus c'$ and e''' the one in $\bar{c}' \smallsetminus \bar{c}$. We have $c \cup \{e\} \cup \{e''\} = \bar{c}' = c \cup \{e'''\} \cup \{e'\}$. Thus, it must be

$$(e = e''' \text{ and } e'' = e') \text{ or } (e = e' \text{ and } e''' = e'').$$

Now, since $e\ co\ e'$, it cannot be $e = e'$ and we must discard the second hypothesis. Therefore, $e = e'''$, i.e., $(c' \smallsetminus c) = (\bar{c}' \smallsetminus \bar{c})$ (and necessarily $(\bar{c} \smallsetminus c) = (\bar{c}' \smallsetminus c')$).

($\Leftarrow$). First suppose $c \subseteq \bar{c}$. Since then event e in $(c' \smallsetminus c) = (\bar{c}' \smallsetminus \bar{c})$ is enabled both in c and $\bar{c}$, it means that for any $\bar{e} \in (\bar{c} \smallsetminus c)$ we have $\bar{e}\ co\ e$. Moreover, we can order the events in $\bar{c} \setminus c$ in a chain $\bar{e}_0 \cdots \bar{e}_n$ in a such a way that $c \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\} \vdash \bar{e}_i$, for $i = 0, \ldots, n$. To this aim, it is enough to choose at each step i one of the maximal events in $(\bar{c} \setminus c) \setminus \{\bar{e}_0, \ldots, \bar{e}_{i-1}\}$ with respect to the $\leq_{ES}$ order.

Now, since $\bar{e}_i$ *co* e, for each $i = 0, \ldots, n$ there exists a diamond

$$\begin{tikzcd}
 & c \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\} \arrow[dl, "a"'] \arrow[dr, "b"] & \\
c \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\} \cup \{e\} \arrow[dr, "b"'] & & c \cup \{\bar{e}_0, \ldots, \bar{e}_i\} \arrow[dl, "a"] \\
 & c \cup \{\bar{e}_0, \ldots, \bar{e}_i\} \cup \{e\} &
\end{tikzcd}$$

Then, for $i = 0, \ldots, n$ we have

$$\big(c \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\}, a, c \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\} \cup \{e\}\big) \prec \\ \big(c \cup \{\bar{e}_0, \ldots, \bar{e}_i\}, a, c \cup \{\bar{e}_0, \ldots, \bar{e}_i\} \cup \{e\}\big),$$

i.e., $(c, a, c') \sim (\bar{c}, a, \bar{c}')$.

To complete the proof, consider $\bar{c} \cap c$. Necessarily, it enables e. So, we have that $\big((\bar{c} \cap c), a, (\bar{c} \cap c) \cup \{e\}\big) \in \mathit{Tran}$. Since $(\bar{c} \cap c) \subseteq \bar{c}$ and $(\bar{c} \cap c) \subseteq c$, from the previous part of the proof we have, $(c, a, c') \sim \big((\bar{c} \cap c), a, (\bar{c} \cap c) \cup \{e\}\big) \sim (\bar{c}, a, \bar{c}')$. ✓

Exploiting Lemma 4.11, it is easy to show the following proposition.

Proposition 4.12
The transition system $les.otsi(ES)$ is an occurrence transition system with independence.

Proof. We verify only the property of occurrence transition systems with independence. Suppose that $(c', b, c) \neq (c'', a, c) \in \mathit{Tran}$. Then, we have $c = c' \cup \{e'\} = c'' \cup \{e''\}$. Since $c' \neq c''$, it must be $e' \neq e''$. Moreover, it is $e' \not\# e''$, since both events appear in c. It cannot be $e' < e''$ nor $e'' < e'$, because otherwise either c' or c'' would not be a configuration. So, it is e' *co* e''. It follows that $\bar{c} = c' \setminus \{e'\} = c'' \setminus \{e''\}$ is a configuration such that $\mathit{Diam}_{a,b}(\bar{c}, c', c'', c)$. ✓

Let us define the opposite transformation from $\underline{\mathsf{oTSI}}$ to $\underline{\mathsf{LES}}$. For $OTI = (S, s^I, L, \mathit{Tran}, I)$ an occurrence transition system with independence, define $otsi.les(OTI)$ to be the structure $(\mathit{Tran}_\sim, \leq, \#, \ell, L)$ where, writing $(s, a, s') \in \pi$ to mean that (s, a, s') occurs in the path π,

- $\mathit{Tran}_\sim$ is the set of the $\sim$-equivalence classes of Tran;
- $[(s, a, s')]_\sim < [(\bar{s}, b, \bar{s}')]_\sim$ if and only if

  $$\forall \pi(\underline{\bar{s}}, b, \underline{\bar{s}}') \in \mathit{Path}(OTI) \text{ with } (\underline{\bar{s}}, b, \underline{\bar{s}}') \sim (\bar{s}, b, \bar{s}'), \\ \exists (\underline{s}, a, \underline{s}') \sim (s, a, s') \text{ such that } (\underline{s}, a, \underline{s}') \in \pi,$$

  and $\leq$ is the reflexive closure of $<$;

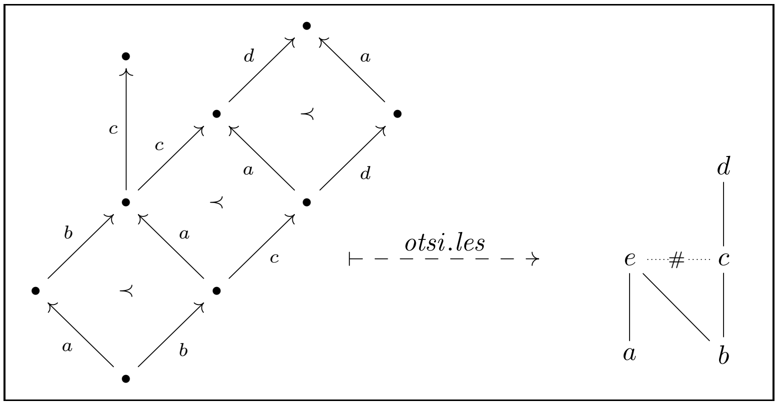


Figure 2: An occurrence transition system OTI and $otsi.les(OTI)$.

- $[(s,a,s')]_\sim \;\#\; [(\bar{s},b,\bar{s}')]_\sim$ if and only if

$$\begin{aligned}&\forall \pi \in Path(OTI),\\ &\quad \forall(\underline{\bar{s}},b,\underline{\bar{s}}') \sim (\bar{s},b,\bar{s}') \text{ and } \forall(\underline{s},a,\underline{s}') \sim (s,a,s')\\ &\qquad\qquad (\underline{s},a,\underline{s}') \in \pi \quad \text{implies} \quad (\underline{\bar{s}},a,\underline{\bar{s}}') \notin \pi;\end{aligned}$$

- $\ell\big([(s,a,s')]_\sim\big) = a$.

It is easy to see that $otsi.les(OTI)$ is a labelled event structure. Figure 2 shows an example of the labelled event structure associated to an occurrence transition system with independence.

Next, we need to extend $otsi.les$ to a functor. Given $(\sigma,\lambda)\colon OTI_0 \to OTI_1$, define $otsi.les\big((\sigma,\lambda)\big) = (\eta_\sigma,\lambda)$, where

$$\eta_\sigma\big([(s,a,s')]_\sim\big) = \begin{cases} \big[\big(\sigma(s),\lambda(a),\sigma(s')\big)\big]_\sim & \text{if } \lambda{\downarrow}a\\ \uparrow & \text{otherwise.}\end{cases}$$

In the proof of Proposition 4.8, it has been shown that $(s,a,s') \prec (\bar{s},a,\bar{s}')$ and $\lambda{\downarrow}a$ implies $\big(\sigma(s),\lambda(a),\sigma(s')\big) \sim \big(\sigma(\bar{s}),\lambda(a),\sigma(\bar{s}')\big)$. Then η_σ is well-given.

Proposition 4.13

Given a transition system with independence morphism $(\sigma,\lambda)\colon OTI_0 \to OTI_1$, $otsi.les\Big((\sigma,\lambda)\Big)\colon otsi.les(OTI_0) \to otsi.les(OTI_1)$ is a labelled event structure morphism.

Proof. We show the properties of labelled event structure morphisms.

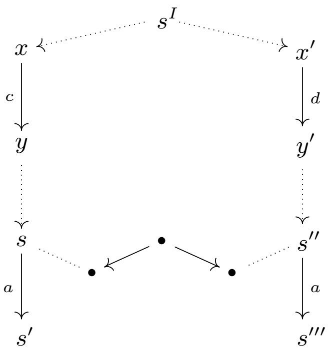
i) $\lfloor \eta_\sigma(e) \rfloor \subseteq \eta_\sigma(\lfloor e \rfloor)$.
Consider $[(\bar{s},b,\bar{s}')]_\sim < \eta_\sigma(e)$ in $otsi.les(OTI_1)$. For each path $\pi_s(s,a,s')$ in OTI_0 with $(s,a,s') \in e$, since its image via (σ,λ) ends with $\big(\sigma(s),\lambda(a),\sigma(s')\big) \in \eta_\sigma(e)$, there must be a transition $(x,c,y) \in \pi_s$ such that $\big(\sigma(x),\lambda(c),\sigma(y)\big) \sim (\bar{s},b,\bar{s}')$, i.e., $\eta_\sigma\big([(x,c,y)]_\sim\big) = [(\bar{s},b,\bar{s}')]_\sim$. We need to prove that $[(x,c,y)]_\sim < [(s,a,s')]_\sim$,

which reduces to prove that, for $\pi_s(s,a,s')$ and $\pi_{s''}(s'',a,s''')$ generic paths as above, letting (x,c,y) and (x',d,y') denote respectively the transitions of π_s and $\pi_{s''}$ mapped to transitions $\sim$-equivalent to $(\bar{s},b,\bar{s}')$, we have

$$(x,c,y) \sim (x',d,y').$$

First observe that, since $\big(\sigma(x'),\lambda(d),\sigma(y')\big) \sim \big(\sigma(x),\lambda(c),\sigma(y)\big)$, no more than one element of $[(x',d,y')]_\sim \cup [(x,c,y)]_\sim$ can appear on the same path, for otherwise, taking the image of such a path via (σ,λ), we would find a path of OTI_1 with more than one occurrence of elements from $\Big[\big(\sigma(x),\lambda(c),\sigma(y)\big)\Big]_\sim$.

Now suppose $(x',d,y') \not\sim (x,c,y)$. Then we are in the situation illustrated by the figure. Necessarily, it must exist



$$(\bar{x},c,\bar{y}) \sim (x,c,y)$$

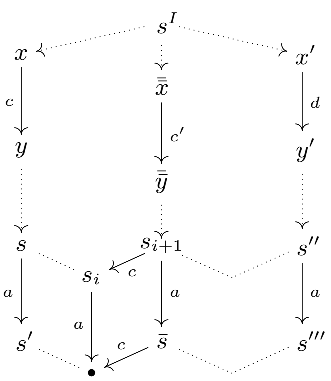
which occurs *'backward'* in the sequence

$$s \leftrightarrow s_1 \leftrightarrow \cdots s_n \leftrightarrow s''.$$

This is because the path from $s^I_{OTI_0}$ to s'' cannot contain any representative of $[(x,c,y)]_\sim$.

So suppose that $s_{i+1} = \bar{x} \xrightarrow{c} \bar{y} = s_i$.

Now take any path $\pi_{s_{i+1}}$, and consider $\pi_{s_{i+1}}(s_{i+1},a,\bar{\bar{s}})$, with $(s_{i+1},a,\bar{\bar{s}}) \sim (s,a,s')$. The situation is illustrated by the figure on the side. Since $\pi_{s_{i+1}}(s_{i+1},a,\bar{\bar{s}})$ is a path whose image via (σ,λ) ends with an element of $\Big[\big(\sigma(s),\lambda(a),\sigma(s')\big)\Big]_\sim$, namely, $\big(\sigma(s_{i+1}),\lambda(a),\sigma(\bar{\bar{s}})\big)$, it follows that $\pi_{s_{i+1}}$ must contain $\bar{\bar{x}} \xrightarrow{c'} \bar{\bar{y}}$ such that



$$\big(\sigma(\bar{\bar{x}}),\lambda(c'),\sigma(\bar{\bar{y}})\big) = (\underline{\bar{\bar{s}}},b,\underline{\bar{\bar{s}}}') \sim (\bar{s},b,\bar{s}').$$

Now consider the path

$$\pi_{s_{i+1}}(s_{i+1},c,s_i) = \pi_{s_{i+1}}(\bar{x},c,\bar{y}).$$

Clearly, its image through (σ,λ) contains

$$\big(\sigma(\bar{\bar{x}}),\lambda(c'),\sigma(\bar{\bar{y}})\big) = (\underline{\bar{\bar{s}}},b,\underline{\bar{\bar{s}}}') \sim (\bar{s},b,\bar{s}')$$

and, in addition, also $\big(\sigma(\bar{x}),\lambda(c),\sigma(\bar{y})\big) \sim \big(\sigma(x),\lambda(c),\sigma(y)\big) \sim (\underline{\bar{s}},b,\underline{\bar{s}}') \sim (\bar{s},b,\bar{s}')$, where $(\underline{\bar{\bar{s}}},b,\underline{\bar{\bar{s}}}') \neq (\underline{\bar{s}},b,\underline{\bar{s}}')$. This is absurd, because no such path can exist in OTI_1.

It follows that $(x,c,y) \sim (x',d,y')$.

ii) $\eta_\sigma(e) \between \eta_\sigma(e') \Rightarrow e \between e'$.
Observe that if $\eta_\sigma(e) = \eta_\sigma(e')$ or $\eta_\sigma(e) \# \eta_\sigma(e')$, then no more than one element from $e \cup e'$ may occur in the same path. This is because, in such a case, there would be a path in OTI_1 in which more than one representative of the same class

or two representatives of conflicting classes would appear in the same path. From such considerations, it follows that it can be neither $e < e'$ nor $e' < e$ nor $e \ co\ e'$. The only possible cases are, therefore, $e = e'$ or $e \# e'$.

iii) $\lambda\big(\ell_{OTI_0}(e)\big) = \ell_{OTI_1}\big(\eta_\sigma(e)\big)$. Immediate. ✓

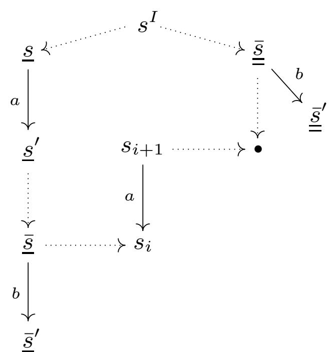
It is very easy now to prove the following result.

Corollary 4.14 ($otsi.les$: $\underline{\mathsf{oTSI}} \to \underline{\mathsf{LES}}$)
The map $otsi.les$ is a functor from $\underline{\mathsf{oTSI}}$ to $\underline{\mathsf{LES}}$.

In order to show that $otsi.les$ and $les.otsi$ form a coreflection, we need the following sequence of lemmas.

Lemma 4.15
Whenever $[(s,a,s')]_\sim\ co\ [(\bar{s},b,\bar{s}')]_\sim$, then $(s,a,s')\ I\ (\bar{s},b,\bar{s}')$.

Proof. By hypothesis $[(s,a,s')]_\sim \not\# [(\bar{s},b,\bar{s}')]_\sim$ and $[(s,a,s')]_\sim \not\lesseqgtr [(\bar{s},b,\bar{s}')]_\sim$. From the first hypothesis, it must exist a path which includes representatives of both classes, say $\pi_{\underline{s}}(\underline{s},a,\underline{s}')\pi_{\underline{\bar{s}}}(\underline{\bar{s}},b,\underline{\bar{s}}')$. Then, from the second condition, it must exist a path which contains a representative of $[(\bar{s},b,\bar{s}')]_\sim$ but no representative of $[(s,a,s')]_\sim$, say $\pi_{\underline{\underline{\bar{s}}}}(\underline{\underline{\bar{s}}},b,\underline{\underline{\bar{s}}}')$.



Now, since no representative of $[(s,a,s')]_\sim$ is in $\pi_{\underline{\underline{\bar{s}}}}$, by Lemma 4.5, there is a sequence

$$\underline{\bar{s}} \leftrightarrow s_1 \leftrightarrow \cdots \leftrightarrow s_n \leftrightarrow \underline{\underline{\bar{s}}}$$

such that there exists $(s_{i+1},a,s_i) \sim (s,a,s')$, as illustrated in the figure. So,

$$(s,a,s') \sim (s_{i+1},a,s_i)\ I\ (\underline{\underline{\bar{s}}},b,\underline{\underline{\bar{s}}}') \sim (\bar{s},b,\bar{s}'),$$

which implies, by the property *(iv)* of transition systems with independence in Definition 3.7, $(s,a,s')\ I\ (\bar{s},b,\bar{s}')$. ✓

Lemma 4.16
Suppose that there is a path $\pi_s(s,a,s')\pi_{\bar{s}}(\bar{s},b,\bar{s}') \in Path(OTI)$ and that, for each $(x,a,y) \in \pi_{\bar{s}}$ we have $[(x,a,y)]_\sim\ co\ [(\bar{s},b,\bar{s}')]_\sim$. Then there exists a transition $(s',b,s'') \in Tran_{OTI}$ such that $(s',b,s'') \sim (\bar{s},b,\bar{s}')$.

Proof. By induction on the length of $\pi_{\bar{s}}$. If $\pi_{\bar{s}}$ is empty there is nothing to show. Otherwise, we have $\pi_s(s,a,s')\pi_{\bar{\bar{s}}}(\bar{\bar{s}},c,\bar{s})(\bar{s},b,\bar{s}')$, where $[(\bar{\bar{s}},c,\bar{s})]_\sim\ co\ [(\bar{s},b,\bar{s}')]_\sim$. So, by the previous lemma, we have $(\bar{\bar{s}},c,\bar{s})\ I\ (\bar{s},b,\bar{s}')$, that, by the general properties of transition systems with independence, must be part of a diamond of concurrency. Therefore, there exists $(\bar{\bar{s}},b,\underline{\bar{s}}) \sim (\bar{s},b,\bar{s}')$ and thus, we have a path $\pi_s(s,a,s')\pi_{\bar{\bar{s}}}(\bar{\bar{s}},b,\underline{\bar{s}})$, where $\pi_{\bar{\bar{s}}}$ is strictly shorter than $\pi_{\bar{s}}$. Then, by induction, there exists (s',b,s'') such that $(s',b,s'') \sim (\bar{\bar{s}},b,\underline{\bar{s}}) \sim (\bar{s},b,\bar{s}')$, which is the thesis. ✓

Lemma 4.17
Consider a path $\pi_s \in Path(OTI)$ and a class $[t]_\sim$ such that

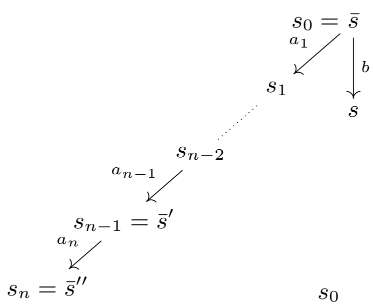
$$\text{for each } t' \text{ in } \pi_s, \text{ we have } [t']_\sim \not\# [t]_\sim \text{ and } [t']_\sim \neq [t]_\sim.$$

Then, there exists $\pi_s \pi_{s'}(s', a, s'') \in Path(OTI)$ with $(s', a, s'') \sim t$.

Proof. By induction on the depth of s, i.e., the length of π_s.

If $\pi_s = \epsilon$, the thesis is trivial, since OTI is reachable. Then, suppose we have $\pi_s = \pi_{\bar{s}}(\bar{s}, b, s)$. By induction hypothesis, there exists a path $\pi_{\bar{s}}\pi_{\bar{s}'}(\bar{s}', a, \bar{s}'')$, with $(\bar{s}', a, \bar{s}'') \sim t$. From the previous lemma, we can assume that $\pi_{\bar{s}'}$ does not contain any transition whose class is concurrent with $[t]_\sim$. In fact, such transitions can be pushed after the representative of $[t]_\sim$. It follows that $\pi_{\bar{s}'}$ contains only elements t' such that $[t']_\sim \leq [t]_\sim$.

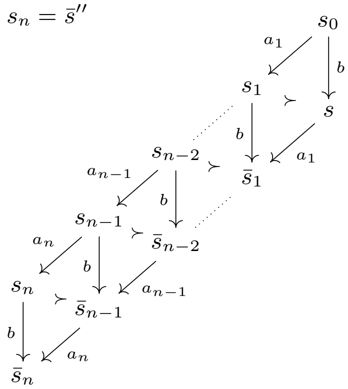
Now, if the first transition of $\pi_{\bar{s}'}$ is $(\bar{s}, b, s)$, we are done. Otherwise, we have the situation shown in the picture on the side, i.e., a chain



$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} s_{n-1} \xrightarrow{a_n} s_n,$$

where $s_0 = \bar{s}$, $s_{n-1} = \bar{s}'$, $s_n = \bar{s}''$, $a_n = a$ and $s = s_i$, for $i = 1, \ldots, n$.

Of course, since $[(s_{i-1}, a_i, s_i)]_\sim \leq [t]_\sim$ for $i = 1, \ldots, n$, and since $[(\bar{s}, b, s)]_\sim \neq [t]_\sim$ and $[(\bar{s}, b, s)]_\sim \not\# [t]_\sim$, we have that, for $i = 1, \ldots, n$, $[(\bar{s}, b, s)]_\sim \not\# [(s_{i-1}, a_i, s_i)]_\sim$, i.e., $(\bar{s}, b, s)\ I\ (s_{i-1}, a_i, s_i)$, for $i = 1, \ldots, n$.



It follows that we can complete the picture as shown in the picture and construct a sequence of diamonds of concurrency. So, we have a path

$$\pi_s(s, a_1, \bar{s}_1) \cdots (\bar{s}_{n-1}, a_n, \bar{s}_n),$$

where $(\bar{s}_{n-1}, a_n, \bar{s}_n) \sim (\bar{s}', a, \bar{s}'') \sim t$, i.e., a path $\pi_s \pi_{s'}(s', a, s'')$ as required. ✓

Lemma 4.18
Consider a path $\pi_s \in Path(OTI)$ and a class $[t]_\sim$ such that

i) for each t' in π_s, it is $[t']_\sim \not\# [t]_\sim$ and $[t']_\sim \neq [t]_\sim$,

ii) for each $[t']_\sim < [t]_\sim$, there exists a representative of $[t']_\sim$ in π_s.

Then, there exists $(s, a, s') \in Tran_{OTI}$ with $(s, a, s') \sim t$.

Proof. By the previous lemma, we find $\pi_s \pi_{\bar{s}'}(\bar{s}', a, \bar{s}'')$ with $(\bar{s}', a, \bar{s}'') \sim t$. Now, consider an element $t' \in \pi_{\bar{s}'}$. We have $[t']_\sim \not\leq [t]_\sim$, because otherwise another representative of $[t']_\sim$ would be in π_s and, by Corollary 4.3, this is impossible. Moreover, $[t]_\sim \not\leq [t']_\sim$, because in the path $\pi_s \pi_{\bar{s}'}(\bar{s}', a, \bar{s}'')$ transition t' occurs before than $(\bar{s}', a, \bar{s}'')$; and it is $[t']_\sim \not\# [t]_\sim$ because in $\pi_s \pi_{\bar{s}'} (\bar{s}', a, \bar{s}'')$ both t' and $(\bar{s}', a, \bar{s}'')$ occur. It follows that $[t']_\sim$ *co* $[t]_\sim$.

Therefore, by applying Lemma 4.16, we find $(s, a, s') \sim (\bar{s}', a, \bar{s}'') \sim t$. ✓

Exploiting the above lemma, we prove next a one-to-one correspondence between the states of OTI and the finite configurations of $otsi.les(OTI)$, or, in other words, states of $les.otsi\big(otsi.les(OTI)\big)$.

Consider the map $\mathcal{C}\colon S_{OTI} \to \mathcal{L}_F\big(otsi.les(OTI)\big)$ given by the correspondence $s \mapsto \big\{[t]_\sim \;\big|\; t \in \pi_s,\ \pi_s \in Path(OTI)\big\}$. Of course, since any path leading to s contains the same equivalence classes, $\mathcal{C}$ is well-defined. Moreover, we have the following easy lemma.

Lemma 4.19

For $s \in S_{OTI}$, the set $\mathcal{C}(s)$ is a finite configuration of $otsi.les(OTI)$.

Let c be a finite configuration of $otsi.les(OTI)$ and let $\varsigma = [t_0]_\sim [t_1]_\sim \cdots [t_n]_\sim$ be a *securing* for c. There is a *unique* path $\pi(\varsigma) = (s_0, a_1, s_1) \cdots (s_{n-1}, a_n, s_n)$ such that $s^I_{OTI} = s_0$, $s_n = s$ and $[(s_{i-1}, a_i, s_i)]_\sim = [t_i]_\sim$, for $i = 1, \ldots, n$. It can be obtained as follows.

- (s_0, a_1, s_1) is the unique element in $[t_0]_\sim$ whose source state is s^I_{OTI}. It exists, by Lemma 4.18, since $\lfloor [t_0]_\sim \rfloor = \varnothing$, and it is unique because of property *(iv)* of Definition 3.7 of transition systems with independence.
- Inductively, (s_{i-1}, a_i, s_i) is the unique element in $[t_i]_\sim$ whose source state is s_{i-1}. Again, it exists because $(s_0, a_1, s_1) \cdots (s_{i-2}, a_n, s_{i-1})$ and $[t_i]_\sim$ satisfy the conditions of Lemma 4.18 and it is unique by definition of transition systems with independence.

It is important to observe that, although the actual path $\pi(\varsigma)$ strictly depends on ς, the state reached does not.

Lemma 4.20

Let c be a finite configuration of otsi.les(OTI) and let $\varsigma = [t_0]_\sim \cdots [t_n]_\sim$ and $\varsigma' = [t'_0]_\sim \cdots [t'_n]_\sim$ be two securings for c. Then the paths $\pi(\varsigma)$ and $\pi(\varsigma')$ obtained as illustrated above reach the same state.

Proof. It is enough to show that $\pi(\varsigma) \simeq \pi(\varsigma')$. To this aim, we work by induction on the *minimal* number n of *'swappings'* of adjacent elements in ς' needed to transform it in ς. Observe that such a number exists since ς and ς' are securing of the same configuration, and, as such, they are just different permutations of the same elements.

If $n = 0$, then $\pi(\varsigma) = \pi(\varsigma')$, since the paths are uniquely determined by the securing. Supposing that we proved the thesis for the case of n swappings, let $\varsigma'' = [t'_0]_\sim \cdots [t'_{i-1}]_\sim [t'_{i+1}]_\sim [t'_i]_\sim [t'_{i+2}]_\sim \cdots [t'_n]_\sim$ be obtained after the first of $n+1$ swappings. Observe that $[t'_{i+1}]_\sim$ must occur in ς before than $[t'_i]_\sim$, otherwise, avoiding the swapping of $[t'_i]_\sim$ and $[t'_{i+1}]_\sim$, we would find a shorter sequence of swappings transforming ς' in ς. It follows that $[t'_i]_\sim \not< [t'_{i+1}]_\sim$, i.e., ς'' is a securing of c. Moreover, $[t'_i]_\sim$ *co* $[t'_{i+1}]_\sim$. Therefore, we have $\pi(\varsigma'') \simeq \pi(\varsigma')$. Now, ς'' can be transformed in ς with n swappings, and therefore, by induction hypothesis, $\pi(\varsigma'') \simeq \pi(\varsigma)$. So, we conclude $\pi(\varsigma) \simeq \pi(\varsigma')$. ✓

Therefore, we can define a map $\mathcal{S}\colon \mathcal{L}_F(otsi.les(OTI)) \to S_{OTI}$ by saying that $c \mapsto s$, where s is the state reached by a path $\pi(\varsigma)$ for a securing ς of c. Now, we can see that $\mathcal{C}$ is an isomorphism of sets whose inverse is $\mathcal{S}$.

Lemma 4.21
$\mathcal{S} = \mathcal{C}^{-1}$.

Proof. Consider $\mathcal{C}(s) = \{[t]_\sim \mid t \in \pi_s\}$ and consider the sequence $\varsigma = [t_0]_\sim \cdots [t_n]_\sim$ such that $\pi_s = t_0 \cdots t_n$. This is clearly a securing of $\mathcal{C}(s)$, whose associated path $\pi(\varsigma)$ is π_s itself. This is because of the uniqueness of $\pi(\varsigma)$ discussed earlier. So, we have $\mathcal{S}(\mathcal{C}(s)) = s$. Suppose $\mathcal{S}(c) = s$. Among the path leading to s, consider $\pi(\varsigma)$, $\varsigma = [t_0]_\sim \cdots [t_n]_\sim$ being a securing of c. Then, we may use $\pi(\varsigma)$ to calculate $\mathcal{C}(\mathcal{S}(c)) = \{[t]_\sim \mid t \in \pi_\varsigma\} = \{[t_i]_\sim \mid i = 0, \ldots, n\} = c$. ✓

It is worthwhile to observe that $\mathcal{C}$ and $\mathcal{S}$ give rise to morphisms of transition systems which are each other's inverse. First observe that $\mathcal{S}(\varnothing) = s^I_{OTI}$, since the unique path associated with the unique securing of the empty configuration, is the empty path. Moreover, $\mathcal{C}(s^I_{OTI}) = \varnothing$, since the unique path leading to s^I_{OTI} in OTI is the empty path. Moreover, we have the following easy lemma.

Lemma 4.22
Let OTI be a transition system with independence. Then

i) If (s, a, s') is a transition of OTI, then $\big(\mathcal{C}(s), a, \mathcal{C}(s')\big)$ is a transition of $les.otsi\big(otsi.les(OTI)\big)$.

ii) If (c, a, c') is a transition of $les.otsi\big(otsi.les(OTI)\big)$, $\big(\mathcal{S}(c), a, \mathcal{S}(c')\big)$ is a transition of OTI.

This means that $(\mathcal{C}, id)$ from OTI to $les.otsi\big(otsi.les(OTI)\big)$ and $(\mathcal{S}, id)$ from $les.otsi\big(otsi.les(OTI)\big)$ to OTI are morphisms of transition systems. Moreover, $(\mathcal{S}, id) = (\mathcal{C}, id)^{-1}$. Recall that $(c, a, c')\ I\ (\bar{c}, b, \bar{c}')$ implies, by definition of $les.otsi$, that $(c' \setminus c) = [t]_\sim\ co\ [\bar{t}]_\sim = (\bar{c}' \setminus \bar{c})$. From the previous Lemma 4.22 we have, therefore, that

$$[t]_\sim = \big[\big(\mathcal{S}(c), a, \mathcal{S}(c')\big)\big]_\sim\ co\ \big[\big(\mathcal{S}(\bar{c}), b, \mathcal{S}(\bar{c}')\big)\big]_\sim = [\bar{t}]_\sim$$

and then, from Lemma 4.15, $\big(\mathcal{S}(c), a, \mathcal{S}(c')\big)\ I\ \big(\mathcal{S}(\bar{c}), b, \mathcal{S}(\bar{c}')\big)$. Therefore we have the following.

Proposition 4.23
$(\mathcal{S}, id)$ is a transition system with independence morphism.

However, $(\mathcal{C}, id)$ is *not* a morphism in TSI. It follows that $(\mathcal{S}, id)$, in general, is *not* an isomorphism of transition systems with independence. Consider now the property:

$$t\ I\ t' \quad \Rightarrow \quad \exists s.\ (s, a, s') \sim t \quad \text{and} \quad (s, b, s'') \sim t'. \tag{E}$$

Proposition 4.24

OTI enjoys property (E) *if and only if* $(\mathcal{C}, id)$ *is a morphism of transition systems with independence.*

Proof. $(\Rightarrow)$. It is enough to show that $(\mathcal{C}, id)$ preserves independence. Suppose $(s, a, s')\ I\ (\bar{s}, b, \bar{s}')$. By condition (E), there exists

$$(s, a, s') \sim (\underline{s}, a, \underline{s}')\ I\ (\underline{s}, b, \underline{s}'') \sim (\bar{s}, b, \bar{s}'),$$

and then, we have $Diam_{a,b}(\underline{s}, \underline{s}', \underline{s}'', u)$. So, we have $[(s, a, s')]_\sim\ co\ [(\bar{s}, b, \bar{s}')]_\sim$. From Lemma 4.22, we have $\mathcal{C}(s') = \mathcal{C}(s) \cup \big\{[(s, a, s')]_\sim\big\}$ and $\mathcal{C}(\bar{s}') = \mathcal{C}(\bar{s}) \cup \big\{[(\bar{s}, b, \bar{s}')]_\sim\big\}$. Therefore, $\big(\mathcal{C}(s), a, \mathcal{C}(s')\big)\ I\ \big(\mathcal{C}(\bar{s}), b, \mathcal{C}(\bar{s}')\big)$.

$(\Leftarrow)$. Suppose that $(\mathcal{C}, id)$ preserves independence. Then $(s, a, s')\ I\ (\bar{s}, b, \bar{s}')$ implies $\big(\mathcal{C}(s), a, \mathcal{C}(s')\big)\ I\ \big(\mathcal{C}(\bar{s}), b, \mathcal{C}(\bar{s}')\big)$, that is $[(s, a, s')]_\sim\ co\ [(\bar{s}, b, \bar{s}')]_\sim$. Then, by repeated applications of Lemma 4.18, we can find a path $\pi_{\underline{s}}(\underline{s}, a, \underline{s}')(\underline{s}', b, u)$ where $(s, a, s,')\sim(\underline{s}, a, \underline{s}')\ I\ (\underline{s}', b, u) \sim (\bar{s}, b, \bar{s},')$. Then, by property *(iii)* of transition system with independence, there exists $\underline{s}''$ and $(\underline{s}, b, \underline{s}'') \sim (\underline{s}', b, u) \sim (\bar{s}, b, \bar{s}')$, i.e., OTI enjoys property (E). ✓

Finally, we can define, for each labelled event structure ES a morphism $(\eta, id)\colon ES \to otsi.les \circ les.otsi(ES)$ as a candidate for the unit of the adjunction. Let us consider η such that

$$\eta(e) = \big[\big(c, a, c \cup \{e\}\big)\big]_\sim.$$

We have already shown in Lemma 4.11 that $(c, a, c') \sim (\bar{c}, a, \bar{c}')$ if and only if $(c' \setminus c) = (\bar{c}' \setminus \bar{c})$. It follows immediately that η is well-defined and is *injective*. Moreover, since any transition of $les.otsi(ES)$, say (c, a, c'), is associated with an event of ES, namely, $c' \setminus c$, we have that η is also *surjective*. Finally, it is not difficult to show that (η, id) is an isomorphism of labelled event structures whose inverse is $(\bar{\eta}, id)$, where $\bar{\eta}\colon [(c, a, c')]_\sim \mapsto (c' \setminus c)$.

Proposition 4.25 ($(\eta, id)\colon ES \to otsi.les \circ les.otsi(ES)$ *is universal*)

For any labelled event structure ES, any occurrence transition system with independence OTI, and any morphism $(\bar{\eta}, \lambda)\colon ES \to otsi.les(OTI)$, there exists a unique k in <u>oTSI</u> *such that $otsi.les(k) \circ (\eta, id) = (\bar{\eta}, \lambda)$.*

$$\begin{tikzcd} ES \arrow[r, "(\eta{,}id)"] \arrow[dr, "(\bar{\eta}{,}\lambda)"'] & otsi.les \circ les.otsi(ES) \arrow[d, "otsi.les(k)"] \\ & otsi.les(OTI) \end{tikzcd}$$

Proof. Let us define $k\colon les.otsi(ES) \to OTI$. Clearly, in order to make the diagram commute, k must be of the form (σ, λ), for some σ. Let us consider $\sigma\colon c \mapsto \mathcal{S}(\bar{\eta}(c))$, i.e.,

$$(\sigma, \lambda) = (\mathcal{S}, id) \circ (\bar{\eta}, \lambda)\colon les.otsi(ES) \to les.otsi(otsi.les(OTI)) \to OTI.$$

Then, we have immediately that σ is well-defined and that (σ, λ) is a transition system with independence morphism.

Now, we must show that the diagram commutes. We need to show that $\eta_\sigma \circ \eta = \eta_\mathcal{S} \circ \eta_{\bar{\eta}} \circ \eta = \bar{\eta}$. Consider $e \in E_{ES}$ and let a be $\ell(e)$. If $\lambda\uparrow a$, then $\bar{\eta}\uparrow a$ and $\eta_{\bar{\eta}}\uparrow a$ and, therefore, both sides of the above equality are undefined. Suppose otherwise that $\lambda\downarrow a$. We have

$$\begin{aligned} e \overset{\eta}{\mapsto} \big[(c, a, c \cup \{e\})\big]_\sim \quad & \overset{\eta_{\bar{\eta}}}{\mapsto} \quad \big[\big(\bar{\eta}(c), \lambda(a), \bar{\eta}(c) \cup \{\bar{\eta}(e)\}\big)\big]_\sim \\ & \overset{\eta_\mathcal{S}}{\mapsto} \quad \big[\big(\mathcal{S}(\bar{\eta}(c)), \lambda(a), \mathcal{S}(\bar{\eta}(c) \cup \{\bar{\eta}(e)\})\big)\big]_\sim \\ & = \quad \big[\big(\sigma(c), \lambda(a), \sigma(c \cup \{e\})\big)\big]_\sim . \end{aligned}$$

Observe that $\big(\bar{\eta}(c), \lambda(a), \bar{\eta}(c) \cup \{\bar{\eta}(e)\}\big)$ belongs to $les.otsi\big(otsi.les(OTI)\big)$ and is associated with the event $\bar{\eta}(e)$ of $otsi.les(OTI)$. Then, from Lemma 4.22, we have that $\big[\big(\mathcal{S}(\bar{\eta}(c)), \lambda(a), \mathcal{S}(\bar{\eta}(c) \cup \{\bar{\eta}(e)\})\big)\big]_\sim = \bar{\eta}(e)$. The last step to prove the universality of (η, id) is to show that k is the unique transition system with independence morphism from $les.otsi(ES)$ to OTI which makes the diagram commute. Let us suppose that there is k' which does so. It must necessarily be $k' = (\sigma', \lambda)$. Observe from the first part of the proof that in order for the diagram to commute, it must be $\eta_{\sigma'}\big(\big[(c, a, c \cup \{e\})\big]_\sim\big) = \big[\big(\sigma'(c), \lambda(a), \sigma'(c \cup \{e\})\big)\big]_\sim = \bar{\eta}(e) = \big[\big(\sigma(c), \lambda(a), \sigma(c \cup \{e\})\big)\big]_\sim$, for any e such that $\lambda\downarrow\ell(e)$. Exploiting this fact, it is easy to show by induction on the cardinality of c that $\sigma' = \sigma$. ✓

Therefore, we have the following theorem.

THEOREM 4.26 ($les.otsi \dashv otsi.les$)
The map $les.otsi$ extends to a functor from $\underline{\mathsf{LES}}$ to $\underline{\mathsf{oTSI}}$ which is left adjoint to $otsi.les$. Since the unit of the adjunction is an isomorphism, the adjunction is a coreflection.

Next, we show now that $(\mathcal{S}, id)$ is the counit of this coreflection. Actually, the task is fairly easy now: by general results in Category Theory [7, chap. IV, pg. 81], the counit of an adjunction can be determined through the unit as the unique morphism $\varepsilon\colon otsi.les \circ les.otsi(OTI) \to OTI$ which makes the following diagram commute.

$$\begin{tikzcd} otsi.les(OTI) \arrow[r, "{(\eta, id)}"] \arrow[dr, "{(id, id)}"'] & otsi.les \circ les.otsi \circ otsi.les(OTI) \arrow[d, "otsi.les(\varepsilon)"] \\ & otsi.les(OTI) \end{tikzcd}$$

However, in the proof of Proposition 4.25, we have identified a general way to find ϵ. From it we obtain $\epsilon = (\mathcal{S}, id) \circ (id, id)$, which is $(\mathcal{S}, id)$.

The results we have shown earlier about $(\mathcal{S}, id)$ make it easy to identify the full subcategory of $\underline{\mathsf{oTSI}}$ and, therefore, of $\underline{\mathsf{TSI}}$ which is *equivalent* to $\underline{\mathsf{LES}}$, i.e., the category of those transition systems with independence which are (representations of) labelled event structure. Such a result gives yet another characterisation of (the finite elements of) *coherent, finitary, prime algebraic domains* [10, 21]. Moreover, this axiomatisation is given only in terms of conditions on the structure of transition systems.

By general results in Category Theory [7, chap. IV, pg. 91], an equivalence of categories is an adjunction whose unit and counit are both isomorphisms, i.e., which is both a reflection and a coreflection. Then, Proposition 4.24 gives us a candidate for the category of occurrence transition system with independence equivalent to $\underline{\mathsf{LES}}$: we consider $\underline{\mathsf{oTSI}}_{\mathsf{E}}$, the full subcategory of $\underline{\mathsf{oTSI}}$ consisting of those occurrence transition systems with independence satisfying condition (E). To obtain the result, it is enough to verify that $les.otsi\colon \underline{\mathsf{LES}} \to \underline{\mathsf{oTSI}}$ actually lands in $\underline{\mathsf{oTSI}}_{\mathsf{E}}$. In fact, this guarantees that the adjunction $les.otsi \dashv otsi.les\colon \underline{\mathsf{LES}} \rightharpoonup \underline{\mathsf{oTSI}}$ restricts to an adjunction $\underline{\mathsf{LES}} \rightharpoonup \underline{\mathsf{oTSI}}_{\mathsf{E}}$ whose unit and counit are again, respectively, (η, id) and $(\mathcal{S}, id)$, which are isomorphisms. It follows then, that $\underline{\mathsf{oTSI}}_{\mathsf{E}} \cong \underline{\mathsf{LES}}$.

Proposition 4.27
The occurrence transition system with independence $les.otsi(ES)$ satisfies condition (E).

Proof. Suppose $(c, a, c')\ I\ (\bar{c}, b, \bar{c}')$ and let $(c' \smallsetminus c) = \{e\}$ and $(\bar{c}' \smallsetminus \bar{c}) = \{\bar{e}\}$. Then, we have $e\ co\ \bar{e}$. It follows that $\underline{c} = (\lfloor e \rfloor \smallsetminus \{e\}) \cup (\lfloor \bar{e} \rfloor \smallsetminus \{\bar{e}\})$ is a finite configuration of ES which enables both e and $\bar{e}$. Then, $(c, a, c') \sim (\underline{c}, a, \underline{c} \cup \{e\})\ I\ (\underline{c}, b, \underline{c} \cup \{\bar{e}\}) \sim (\bar{c}, b, \bar{c}')$ in $les.otsi(ES)$. ✓

Thus we have the following.

Corollary 4.28
The categories $\underline{\mathsf{LES}}$ and $\underline{\mathsf{oTSI}}_{\mathsf{E}}$ are equivalent.

We can interpret such a result as a demonstration of the claim that transition systems with independence are a generalisation of labelled event structures to a model system. However, the fact that just unfolding transition systems to their occurrence version does not suffice to get a category equivalent to $\underline{\mathsf{LES}}$, stresses that the *independence* relation on transitions is *not* exactly a *concurrency* relation. As an intuitive explanation of this phenomenon, it is very easy to think of a transition system with independence in which independent transitions never occur in the same path, i.e., intuitively, they are in conflict. In the light of such observation, condition (E) can be seen exactly as the condition which guarantees that independence *is* concurrency. It is then that the simple unfolding of transition systems with independence yields the category $\underline{\mathsf{oTSI}}_{\mathsf{E}}$ equivalent to $\underline{\mathsf{LES}}$.

To conclude this section, we briefly see that the coreflection $\underline{\mathsf{LES}} \hookrightarrow \underline{\mathsf{oTSI}}$ cuts down to a coreflection $\underline{\mathsf{dLES}} \hookrightarrow \underline{\mathsf{dTSI}}$,which composes with the coreflection given earlier in this section to give a coreflection $\underline{\mathsf{dLES}} \hookrightarrow \underline{\mathsf{dTSI}}$. As a consequence, we have that $\underline{\mathsf{dLES}} \cong \underline{\mathsf{doTSI}}_{\mathsf{E}}$. These results are shown by the following proposition.

Proposition 4.29
If ES is deterministic, then $les.otsi(ES)$ is deterministic. If OTI is deterministic, then $otsi.les(OTI)$ is deterministic.

Proof. If $(c, a, c \cup \{e\})$ and $(c, b, c \cup \{\bar{e}\})$ are transitions of $les.otsi(ES)$, then $c \vdash e$ and $c \vdash \bar{e}$, and then $a \neq b$.

Suppose that $c \vdash [(s, a, s')]_\sim$ and $c \vdash [(\bar{s}, b, \bar{s}')]_\sim$. Clearly, we can assume c finite. Then, $\big(c, a, c \cup \{[(s, a, s')]_\sim\}\big)$, $\big(c, b, c \cup \{[(\bar{s}, b, \bar{s}')]_\sim\}\big)$ are in $les.otsi\big(otsi.les(OTI)\big)$ and, therefore, $\big(\mathcal{S}(c), a, \mathcal{S}(c \cup \{[(s, a, s')]_\sim\})\big)$, $\big(\mathcal{S}(c), b, \mathcal{S}(c \cup \{[(\bar{s}, b, \bar{s}')]_\sim\})\big)$ are in OTI. Then $a \neq b$. ✓

These results are summarised in the following theorem.

Theorem 4.30 (*Moving along the 'behaviour/system' axis*)

$$
\begin{tikzcd}
 & \underline{\mathsf{TSI}} & & \underline{\mathsf{TS}} \arrow[ll, hook'] \\
\underline{\mathsf{dTSI}} & & \underline{\mathsf{dTS}} \arrow[ll, hook] \arrow[ur, hook] & \\
 & \underline{\mathsf{LES}} \arrow[uu, hook] & & \underline{\mathsf{ST}} \arrow[uu, hook] \arrow[ll, hook] \\
\underline{\mathsf{dLES}} \arrow[uu, hook] & & \underline{\mathsf{HL}} \arrow[ll, hook] \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

5 Deterministic Transition Systems with Independence

Now, we consider the relationship between $\underline{\mathsf{dTSI}}$ and $\underline{\mathsf{TSI}}$, looking for a generalisation of the reflection $\underline{\mathsf{dTS}} \hookrightarrow \underline{\mathsf{TS}}$ in order to provide an 'abstraction functor' from transition systems with independence to a linear time framework. Of course, the question to be answered is whether a left adjoint for the inclusion functor $\underline{\mathsf{dTSI}} \hookrightarrow \underline{\mathsf{TSI}}$ exists or not. Although the answer is positive, it turns out that this is actually a rather complicated issue.

At a first sight, one could be tempted to refine the construction given in case of transition systems by defining a suitable independence relation on the deterministic transition system obtained in that way. However, this would not work, since, in general, no independence relation yields a transition system with independence. Let us see what happens with the following example.

Example 5.1
Consider the transition system T in the following figure together with its deter-

ministic version $ts.dts(T)$.

$$\begin{tikzcd}[row sep=small]
s \arrow[dd, "a"'] \arrow[dr, "a"] & & & & {[s]_\simeq} \arrow[dd, "a"] \\
& s'' \arrow[dl, "c"] & & & \\
s' \arrow[dd, "b"'] & & {} \arrow[r, dashed, mapsto, "ts.dts"] & {} & {[s']_\simeq} \arrow[loop left, "c"] \arrow[dd, "b"] \\
& & & & \\
u & & & & u
\end{tikzcd}$$

Now, suppose that $(s, a, s'')\ I\ (s', b, u)$. Observe, that, in order to establish the reflection *at the level of transition systems with independence, since the unit would be a morphism from the original transition system to the deterministic one, independence must be preserved. Therefore, whatever the independence relation on the deterministic transition system is, it must certainly be $([s]_\simeq, a, [s']_\simeq)\ I\ ([s']_\simeq, b, [u]_\simeq)$. Then, we do not have a transition system with independence, since axiom* (iii) *fails.*

However, in the rest of this section, we will show that it is always possible to 'complete' the deterministic transition system obtained by $ts.dts$ in order to make it a transition system with independence. Moreover, such a completion will be 'universal', so that it will give the reflection we are seeking. In the case of the transition system above, the resulting transition system is shown below.

$$\begin{tikzcd}
{[s]_\simeq} \arrow[d, "a"'] \arrow[dr, "b"] & \\
{[s']_\simeq} \arrow[loop left, "c"] \arrow[dr, "b"'] & x \arrow[d, "a"] \\
& {[u]_\simeq}
\end{tikzcd}$$

Observe that it may also not be possible to define I to be irreflexive. This happens when in the original transition system with independence there are diamonds of concurrency whose transitions carry the same label, for these, when 'collapsed' by the deterministic construction, become *autoindependent*, i.e., independent of themselves. It is easy to realise that the only way to cope with such transitions is by eliminating them from the transition system. In other words, *autoconcurrency*, i.e., concurrency between events carrying the same label, add a further level of difficulty to the problem, since it causes autoindependence in the deterministic transition system.

Definition 5.2 (*Pre-Transition Systems with Independence*)
A pre-transition system with independence *is a transition system together with a binary and symmetric relation I on its transitions.*

A morphism *of pre-transition systems with independence is a transition system morphism which, in addition, preserve the relation I.*

Let <u>pTSI</u> *denote the category of pre-transition systems with independence.*

Given sets S and L, consider triples of the kind $(X, \equiv, I)$, where $X \subseteq S \cdot L^* = \{s\alpha \mid s \in S \text{ and } \alpha \in L^*\}$, and $\equiv$ and I are binary relations on X. On such triples, the following closure properties can be considered.

$$
\begin{array}{lll}
\textit{(Cl1)} & x \equiv z \text{ and } za \in X \quad \text{implies} \quad xa \in X \text{ and } xa \equiv za; \\
\textit{(Cl2)} & x \equiv z \text{ and } za \; I \; yc \quad \text{implies} \quad xa \; I \; yc; \\
\textit{(Cl3)} & xab \equiv xba \text{ and } xa \; I \; xb \text{ or } xa \; I \; xab \\
 & \qquad\qquad\qquad\qquad \text{implies} \quad xa \; I \; yc \Leftrightarrow xba \; I \; yc.
\end{array}
$$

We say that $(X, \equiv, I)$ is *suitable* if $\equiv$ is an *equivalence* relation, I is a *symmetric* relation and it *enjoys* properties *(Cl1)*, *(Cl2)* and *(Cl3)*. Suitable triples are meant to represent deterministic (pre) transition systems with independence, the elements in X representing both states and transitions. Namely, xa represents the state reached from (the state corresponding to) x with an a-labelled transition, and that transition itself. Thus, equivalence $\equiv$ relate paths which lead to the same state and relation I expresses independence of transitions. With this understanding, *(Cl1)* means that from any state there is at most one a-transition, while *(Cl2)* says that I acts on transitions rather than on their representation. Finally, *(Cl3)*—the analogous of axiom *(iv)* of transition systems with independence—tells that transitions on the opposite edges of a diamond behave the same with respect to I.

For $x \in S \cdot L^*$ and $a \in L$, let $x\lceil a$ denote the pruning of x with respect to a. Formally,

$$
s\lceil a = s \qquad \text{and} \qquad (xb)\lceil a = \begin{cases} x\lceil a & \text{if } a = b \\ (x\lceil a)b & \text{otherwise} \end{cases}
$$

Of course, $(x\lceil a)\lceil b = (x\lceil b)\lceil a$ and thus it is possible to use unambiguously $x\lceil A$ for $A \subseteq L$. Given $X \subseteq S \cdot L^*$, we use $X\lceil A$ to denote the set $\{x\lceil A \mid x \in X\}$ whilst, for R a binary relation on X, $R\lceil A$ stands for $\{(x\lceil A, y\lceil A) \mid (x, y) \in R\}$.

For a transition system with independence $TI = (S, s^I, L, \mathit{Tran}, I)$, we define the sequence of triples $(S_i, \equiv_i, I_i)$, for $i \in \omega$, inductively as follows. For $i = 0$, $(S_0, \equiv_0, I_0)$ is the least (with respect to componentwise set inclusion) *suitable* triple such that

$$
S \cup \left\{ sa \;\middle|\; (s, a, u) \in \mathit{Tran} \right\} \subseteq S_0; \quad \left\{ (sa, u) \;\middle|\; (s, a, u) \in \mathit{Tran} \right\} \subseteq \equiv_0;
$$
$$
\left\{ (sa, s'b) \;\middle|\; (s, a, u) \; I \; (s', b, u') \right\} \subseteq I_0;
$$

and, for $i > 0$, $(S_i, \equiv_i, I_i)$ is the least *suitable* triple such that

$$
\begin{array}{ll}
(\Im) & S_{i-1}\lceil A_{i-1} \subseteq S_i; \quad \equiv_{i-1}\lceil A_{i-1} \subseteq \equiv_i; \quad (I_{i-1} \setminus TA_{i-1})\lceil A_{i-1} \subseteq I_i; \\
\textit{(D1)} & xa,\ xb \in S_{i-1}\lceil A_{i-1} \text{ and } xa \; (I_{i-1} \setminus TA_{i-1})\lceil A_{i-1} \; xb \\
 & \qquad\qquad \text{implies} \quad xab,\ xba \in S_i \text{ and } xab \equiv_i xba; \\
\textit{(D2)} & xa,\ xab \in S_{i-1}\lceil A_{i-1} \text{ and } xa \; (I_{i-1} \setminus TA_{i-1})\lceil A_{i-1} \; xab \\
 & \qquad\qquad \text{implies} \quad xb,\ xba \in S_i \text{ and } xab \equiv_i xba;
\end{array}
$$

where $A_i = \{a \in L \mid xa\ I_i\ xa\}$ and $TA_i = \{(xa, yb) \in I_i \mid a \in A_i \text{ or } b \in A_i\}$.

The inductive step extends a triple towards a transition system with independence by means of the rules *(D1)* and *(D2)*, whose intuitive meaning is clearly that of closing possibly incomplete diamonds. The process could create *autoindependent* transitions, namely the transitions with labels in A_{i-1}, which must be eliminated. This is done by *($\Im$)* which removes them from S_i, $\equiv_i$, and I_i.

A simple inspection of the rules shows that if $a \in A_i$, then it will never appear again in the sequence. Thus, if x is removed from S_i, it will not be reintroduced, and the same applies to the pairs in $\equiv_i$ and I_i. Then, it is easy to identify the *limit* of the sequence as

$$\Big(S_\omega = \bigcup_{i\in\omega}\bigcap_{j\geq i} S_j, \quad \equiv_\omega = \bigcup_{i\in\omega}\bigcap_{j\geq i} \equiv_j, \quad I_\omega = \bigcup_{i\in\omega}\bigcap_{j\geq i} I_j\Big).$$

Proposition 5.3
The triple $(S_\omega, \equiv_\omega, I_\omega)$ is suitable. Moreover, I_ω is irreflexive.

Proof. Easy. ✓

The following proposition gives an easy-to-prove alternative characterisation of $(S_\omega, \equiv_\omega, I_\omega)$ which will be useful later on. In the following let A_ω denote $\bigcup_{i\in\omega} A_i$ and let TA_ω be $\bigcup_{i\in\omega} TA_i$.

Proposition 5.4
$(S_\omega, \equiv_\omega, I_\omega) = \big(\bigcup_{i\in\omega}(S_i{\restriction}A_\omega), \quad \bigcup_{i\in\omega}(\equiv_i{\restriction}A_\omega), \quad \bigcup_{i\in\omega}((I_i \smallsetminus TA_\omega){\restriction}A_\omega)\big)$.

In the following we shall refer to the sets obtained by applying rules *($\Im$)*, *(D1)* and *(D2)* to S_{i-1}, $\equiv_{i-1}$ and I_{i-1} as the *generators* of the suitable triple $(S_i, \equiv_i, I_i)$. Similarly, sets $S \cup \{sa \mid (s,a,u) \in \mathit{Tran}\}$, $\{(sa, u) \mid (s,a,u) \in \mathit{Tran}\}$ and $\{(sa, s'b) \mid (s,a,u)\ I\ (s',b,u')\}$ are the generators of $(S_0, \equiv_0, I_0)$. We shall denote the generators of $(S_i, \equiv_i, I_i)$ by ${}_\gamma S_i$, ${}_\gamma{\equiv_i}$ and ${}_\gamma I_i$.

If TI is deterministic then there is a neat characterisation of $(S_0, \equiv_0, I_0)$.

Lemma 5.5
Let TI be a deterministic transition system with independence. Then

i) $s\alpha \equiv_0 s'\beta$ if and only if there is $u \in S$ and two sequences of transitions leading from s to u with labels α and from s' to u with labels β;

ii) $s' \equiv_0 sa$ if and only if $(s,a,s') \in \mathit{Tran}$.

iii) $sa\ I_0\ s'b$ if and only if there exist $(s,a,u)\ I\ (s',b,u')$ in TI.

Proof. Observe that point *(ii)* is an easy corollary of point *(i)*.

Consider $X \subseteq S \cdot L^*$ such that $s\alpha \in X$ if and only if $s \in S$ and there is a sequence of transitions $(s, a_0, s_0)\cdots(s_{n-1}, a_n, s_n)$ in TI, where $a_0 \cdots a_n$ is α. Then, consider the relations $\equiv\ \subseteq X \times X$ and $\bar{I} \subseteq X \times X$ such that $s\alpha \equiv s'\beta$ if and only if the two corresponding sequences of transitions lead to the same state of TI and $s\alpha\ \bar{I}\ s'\beta$ if and only if the last transitions of such sequences are in the relation I of TI.

In order to show *(i)* and *(iii)* it clearly suffices to show that $(X, \equiv, \bar{I}) = (S_0, \equiv_0, I_0)$. To this purpose, one first shows by induction on the structure of the elements of X that $({}_\gamma S_0, {}_\gamma{\equiv_0}, {}_\gamma I_0) \subseteq (X, \equiv, \bar{I}) \subseteq (S_0, \equiv_0, I_0)$. Then, since $(S_0, \equiv_0, I_0)$ is the least *suitable* triple which contains ${}_\gamma S_0$, ${}_\gamma{\equiv_0}$ and ${}_\gamma I_0$, the proof is easily concluded by showing that $(X, \equiv, \bar{I})$ is *suitable*. ✓

This result admits the following immediate corollary.

Corollary 5.6
If TI is deterministic, for any $x \in S_0$ there is exactly one $s \in S$ such that $x \equiv_0 s$.

As anticipated before, $(S_i, \equiv_i, I_i)$ encodes a deterministic pre-transition system with independence which contains a deterministic version of the original TI we started from (apart from the autoindependent transitions). Formally, for each $\kappa \in \omega \cup \{\omega\}$, define

$$TSys_\kappa = \Big(S_\kappa/{\equiv_\kappa}, [s^I]_{\equiv_\kappa}, L_\kappa, Tran_{\equiv_\kappa}, I_{\equiv_\kappa}\Big),$$

where

- $([x]_{\equiv_\kappa}, a, [x']_{\equiv_\kappa}) \in Tran_{\equiv_\kappa}$ if and only if $x' \equiv_\kappa xa$;
- $([x]_{\equiv_\kappa}, a, [x']_{\equiv_\kappa})\ I_{\equiv_\kappa}\ ([\bar{x}]_{\equiv_\kappa}, b, [\bar{x}']_{\equiv_\kappa})$ if and only if $xa\ I_\kappa\ \bar{x}b$;
- $L_\kappa = L \setminus \bigcup_{j<\kappa} A_j$.

Observe that $TSys_\kappa$ is well defined. In fact, concerning $Tran_{\equiv_\kappa}$, since $xa \in S_i$ if and only if $\underline{x}a \in S_i$ for any $\underline{x} \equiv_i x$, and since $x' \equiv_i xa$ if and only if $\underline{x}' \equiv_i \underline{x}a$ for any $\underline{x} \equiv_i x$ and $\underline{x}' \equiv_i x'$, its definition is irrespective of the chosen representative. The same holds for the definition of $I_{\equiv_\kappa}$, since $xa\ I_i\ x'b$ if and only if $\underline{x}a\ I_i\ \underline{x}'b$ for any $\underline{x} \equiv_i x$ and $\underline{x}' \equiv_i x'$.

Proposition 5.7
$TSys_\kappa$ is a deterministic pre-transition system with independence.

Proof. $TSys_\kappa$ is certainly a transition system and since $(S_\kappa, \equiv_\kappa, I_\kappa)$ is *suitable*, $I_{\equiv_\kappa}$ is symmetric. Moreover, since $[x]_{\equiv_\kappa} \xrightarrow{a} [x']_{\equiv_\kappa}$ if and only if $x' \equiv_\kappa xa$, then if $[x]_{\equiv_\kappa} \xrightarrow{a} [x'']_{\equiv_\kappa}$, we have $[x'']_{\equiv_\kappa} = [x']_{\equiv_\kappa}$. Therefore, $TSys_\kappa$ is deterministic. ✓

Lemma 5.5, its Corollary 5.6 and the previous proposition show the similarity of $TSys_0$ with the construction of the deterministic version of a transition system as given in Section 2. Actually, starting from them, it is not difficult to see that, when applied to a transition system TS, i.e., a transition system with independence whose independence relation is empty, $TSys_0$ is a deterministic transition system isomorphic to $ts.dts(TS)$. This fact supports our claim that the construction we are about to give builds on $ts.dts$. However, in Section 2 a simpler construction was enough, because we did not need to manipulate transitions but only states.

Proposition 5.8
The pair (in, id), where $in\colon S \to S_0/{\equiv_0}$ is the function which sends s to its equivalence class $[s]_{\equiv_0}$ and id is the identity of L, is a morphism of pre-transition systems with independence from TI to $TSys_0$. Moreover, if TI is deterministic, then (in, id) is an isomorphism.

Proof. Since $(s, a, s') \in Tran$ implies that $s' \equiv_0 sa$ which in turn implies that $([s]_{\equiv_0}, a, [s']_{\equiv_0}) \in Tran_{\equiv_0}$, we have that (in, id) is a morphism of transition systems. If TI is deterministic then from Corollary 5.6 and from Lemma 5.5 *(ii)*, $(s, a, s') \in Tran$ if and only if $([s]_{\equiv_0}, a, [s']_{\equiv_0}) \in Tran_{\equiv_0}$, and thus (in, id) is an isomorphism of transition systems. Moreover, since $(s, a, s')\ I\ (\bar{s}, b, \bar{s}')$ implies $sa\ I_0\ \bar{s}b$, which in turn implies $([s]_{\equiv_0}, a, [s']_{\equiv_0})\ I_{\equiv_0}\ ([\bar{s}]_{\equiv_0}, b, [\bar{s}']_{\equiv_0})$, it follows that (in, id) is a morphism of pre-transition systems with independence. Finally, from Lemma 5.5 *(iii)*, if TI is deterministic, then $(s, a, s')\ I\ (\bar{s}, b, \bar{s}')$ if and only if $([s]_{\equiv_0}, a, [s']_{\equiv_0})\ I_{\equiv_0}\ ([\bar{s}]_{\equiv_0}, b, [\bar{s}']_{\equiv_0})$, i.e., (in, id) is an isomorphism of (pre) transition systems with independence. ✓

For $i \in \omega \setminus \{0\}$, consider the pair (in_i, id_i), where $in_i\colon S_{i-1}/{\equiv_{i-1}} \to S_i/{\equiv_i}$ is the function such that $in_i([x]_{\equiv_{i-1}}) = [x{\restriction}A_{i-1}]_{\equiv_i}$ and $id_i\colon L_{i-1} \rightharpoonup L_i$ is given by $id_i(a) = a$ if $a \notin A_{i-1}$ and $id_i{\uparrow}a$ otherwise. Then, we have the following.

Lemma 5.9
The pair $(in_i, id_i)\colon TSys_{i-1} \to TSys_i$ is a morphism of pre-transition systems with independence.

Proof. Observe that since $x \equiv_{i-1} y$ implies that $x{\restriction}A_{i-1} \equiv_i y{\restriction}A_{i-1}$, in_i is well defined. We check the conditions in Definition 5.2.

i) $in_i([s^I]_{\equiv_{i-1}}) = [s^I{\restriction}A_{i-1}]_{\equiv_i} = [s^I]_{\equiv_i}$.

ii) Consider a transition $[x]_{\equiv_{i-1}} \xrightarrow{a} [xa]_{\equiv_{i-1}}$ in $TSys_{i-1}$. Now, if $a \in A_{i-1}$, then $in_i([x]_{\equiv_{i-1}}) = [x{\restriction}A_{i-1}]_{\equiv_i} = [xa{\restriction}A_{i-1}]_{\equiv_i} = in_i([xa]_{\equiv_{i-1}})$. Otherwise, $xa{\restriction}A_{i-1} = (x{\restriction}A_{i-1})a$, and then

$$in_i\big([x]_{\equiv_{i-1}}\big) = \big[x{\restriction}A_{i-1}\big]_{\equiv_i} \xrightarrow{a} \big[(x{\restriction}A_{i-1})a\big]_{\equiv_i} = in_i\big([xa]_{\equiv_{i-1}}\big).$$

iii) If $\big([x]_{\equiv_{i-1}}, a, [xa]_{\equiv_{i-1}}\big)\ I_{\equiv_{i-1}}\ \big([y]_{\equiv_{i-1}}, b, [yb]_{\equiv_{i-1}}\big)$ and $a, b \notin A_{i-1}$, then we have $xa\ I_{i-1}\ yb$ and $(x{\restriction}A_{i-1})a\ I_i\ (y{\restriction}A_{i-1})b$, i.e.,

$$\big([x{\restriction}A_{i-1}]_{\equiv_i} \xrightarrow{a} [(x{\restriction}A_{i-1})a]_{\equiv_i}\big)\ I_{\equiv_i}\ \big([y{\restriction}A_{i-1}]_{\equiv_i} \xrightarrow{b} [(y{\restriction}A_{i-1})b]_{\equiv_i}\big),$$

i.e., $\big(in_i([x]_{\equiv_{i-1}}) \xrightarrow{a} in_i([xa]_{\equiv_{i-1}})\big)\ I_{\equiv_i}\ \big(in_i([y]_{\equiv_{i-1}}) \xrightarrow{b} in_i([yb]_{\equiv_{i-1}})\big)$. ✓

It is interesting to notice that $TSys_\omega$ is a colimit in the category $\underline{\mathsf{pTSI}}$.

Proposition 5.10
$TSys_\omega$ is the colimit in $\underline{\mathsf{pTSI}}$ of the $\boldsymbol{\omega}$-diagram

$$\mathcal{D} = TSys_0 \xrightarrow{(in_1, id_1)} TSys_1 \xrightarrow{(in_2, id_2)} \cdots \xrightarrow{(in_i, id_i)} TSys_i \xrightarrow{(in_{i+1}, id_{i+1})} \cdots$$

Proof. The reader is referred to [7, chap. III, pg. 62] for the definition of the categorical concept involved.

For any $i \in \omega$, consider the function $in_i^\omega \colon S_i/{\equiv_i} \to S_\omega/{\equiv_\omega}$ such that $in_i^\omega([x]_{\equiv_i}) = [x\restriction A_\omega]_{\equiv_\omega}$ and let $id_i^\omega \colon L_i \rightharpoonup L_\omega$ denote the function such that $id_i^\omega(a) = a$ if $a \notin A_\omega$ and $id_i^\omega \uparrow a$ otherwise. As for Lemma 5.9, it is easy to see that $(in_i^\omega, id_i^\omega)$ is a morphism of pre-transition systems with independence from $TSys_i$ to $TSys_\omega$.

Since for each $i \in \omega$ we have $in_{i+1}^\omega \circ in_{i+1} = in_i^\omega$ and $id_{i+1}^\omega \circ id_{i+1} = id_i^\omega$, $TSys_\omega$ and the morphisms $\{(in_i^\omega, id_i^\omega) \mid i \in \omega\}$ form a cocone in pTSI with base $\mathcal{D}$. Now, consider any cocone $\{(\sigma_i, \lambda_i) \colon TSys_i \to PT \mid i \in \omega\}$, for PT a pre-transition system with independence. Then, by definition of cocone, it must be $\sigma_i = \sigma_{i+1} \circ in_{i+1}$ for each $i \in \omega$, i.e., $\sigma_i([x]_{\equiv_i}) = \sigma_{i+1}([x\restriction A_i]_{\equiv_{i+1}})$, whence it follows easily that for any $x \in S_i$ and $y \in S_j$ such that $x\restriction A_\omega = y\restriction A_\omega$ it must be $\sigma_i([x]_{\equiv_i}) = \sigma_j([y]_{\equiv_j})$. Moreover, again by definition of cocone, it must be $\lambda_i = \lambda_{i+1} \circ id_{i+1}$. This implies that for $a \in L \smallsetminus A_\omega$ we have $\lambda_i(a) = \lambda_{i+1}(a)$ for any $i \in \omega$, while for $a \in A_j$ it must be $\lambda_i \uparrow a$ for any $i \leq j$. In fact, if $a \notin A_\omega$, since $id_{i+1}(a) = a$, it must be $\lambda_i(a) = \lambda_{i+1}(a)$. Suppose instead that $a \in A_j$. Then, $id_{j+1} \uparrow a$ and thus $\lambda_j \uparrow a$. Now, since $id_i(a) = a$ if $i \leq j$, it follows that $\lambda_i \uparrow a$ for any $i \leq j$.

Now, define $(\bar\sigma, \bar\lambda) \colon TSys_\omega \to PT$, where $\bar\sigma([x]_{\equiv_\omega}) = \sigma_i([\bar x]_{\equiv_i})$ for any i and $\bar x \in S_i$ such that $\bar x\restriction A_\omega = x$, and take $\bar\lambda$ to be the restriction of λ_0 to L_w. Exploiting the features of the morphisms (σ_i, λ_i), it is easy to see that $(\sigma_i, \lambda_i) = (\bar\sigma, \bar\lambda) \circ (in_i^\omega, id_i^\omega)$ for each i, and that $(\bar\sigma, \bar\lambda)$ is the unique morphism which enjoys this property. Observe that, in view of Proposition 5.4, $\bar\sigma$ could be equivalently defined by saying that $\bar\sigma([x]_{\equiv_\omega}) = \sigma_i([x]_{\equiv_i})$ for any x such that $x \in S_i$. ✓

Besides enjoying a (co)universal property, $TSys_\omega$ has another property which the reader would have already guessed: it is actually a deterministic transition system with independence.

Proposition 5.11

$TSys_\omega$ is a deterministic transition system with independence.

Proof. Proposition 5.7 shows that $Tsys_\omega$ is a deterministic pre-transition system with independence, while it follows immediately from Proposition 5.3 that $I_{\equiv_\omega}$ is irreflexive. Let us check the axioms of transition systems with independence.

i) Vacuous, since $TSys_\omega$ is deterministic.

ii) Suppose that $([x]_{\equiv_\omega}, a, [x']_{\equiv_\omega})\ I_{\equiv_\omega}\ ([x]_{\equiv_\omega}, b, [x'']_{\equiv_\omega})$. Then, $xa\ I_\omega\ xb$ and, therefore, there exists an index i such that $xa\ I_{i-1}\ xb$, which, in turn, implies that there exist $xab \equiv_i xba \in S_i$. Then, by *(Cl3)*, $xa\ I_i\ xb$ implies $xba\ I_i\ xb$ and $xb\ I_i\ xa$ implies $xab\ I_i\ xa$. Since $a, b \notin A_\omega$ and $x\restriction A_\omega = x$, then we have $xab \equiv_\omega xba$, and $xa\ I_\omega\ xab$ and $xb\ I_\omega\ xba$, which implies that there exists $[xab]_{\equiv_\omega} = [u]_{\equiv_\omega} = [xba]_{\equiv_\omega}$ in $S_\omega/{\equiv_\omega}$ such that $([x]_{\equiv_\omega}, a, [x']_{\equiv_\omega})\ I_{\equiv_\omega}\ ([x']_{\equiv_\omega}, b, [u]_{\equiv_\omega})$, and $([x]_{\equiv_\omega}, b, [x'']_{\equiv_\omega})\ I_{\equiv_\omega}\ ([x'']_{\equiv_\omega}, a, [u]_{\equiv_\omega})$.

iii) Similar to the previous point.

iv) It is enough to show that

$$\big([x]_{\equiv_\omega}, a, [x']_{\equiv_\omega}\big)\ ({\prec} \cup {\succ})\ \big([x'']_{\equiv_\omega}, a, [u]_{\equiv_\omega}\big) I_{\equiv_\omega}\ \big([\bar x]_{\equiv_\omega}, b, [\bar x']_{\equiv_\omega}\big)$$
$$\text{implies } \big([x]_{\equiv_\omega}, a, [x']_{\equiv_\omega}\big)\ I_{\equiv_\omega}\ \big([\bar x]_{\equiv_\omega}, b, [\bar x']_{\equiv_\omega}\big).$$

Suppose that the '$\prec$' case holds. Then, there exists i such that $x' \equiv_i xa$, $x'' \equiv_i xb$, $xa\ I_i\ xb$, $xab \equiv_i u \equiv_i xba$, and $xba\ I_i\ \bar{x}b$. Then, by *(Cl3)*, we have $xa\ I_i\ \bar{x}b$. Then, it is $xa\ I_\omega\ \bar{x}b$, whence it follows that $([x]_{\equiv_\omega}, a, [x']_{\equiv_\omega})\ I_{\equiv_\omega}\ ([\bar{x}]_{\equiv_\omega}, b, [\bar{x}']_{\equiv_\omega})$.

A similar proof shows the case in which '$\succ$' holds. ✓

Thus, $TSys_\omega$ is the deterministic transition system with independence we will associate to the transition system with independence TI. Formally, define the map $dtsi$ from the objects of $\underline{\mathsf{TSI}}$ to the objects of $\underline{\mathsf{dTSI}}$ as $dtsi(TI) = TSys_\omega$. Figure 3 exemplifies the construction in an easy, yet interesting, case.

Consider $TI = (S, s^I, L, Tran, I)$ and $TI' = (S', s'^I, L', Tran', I')$ together with a morphism $(\sigma, \lambda)\colon TI \to TI'$ in $\underline{\mathsf{TSI}}$. In the following let $(S_\kappa, \equiv_\kappa, I_\kappa)$ and $(S'_\kappa, \equiv'_\kappa, I'_\kappa)$, $\kappa \in \omega \cup \{\omega\}$, be the sequences of suitable triples corresponding, respectively, to TI and TI'. Moreover, we shall write A_κ, TA_κ, L_κ, $TSys_\kappa$, A'_κ, TA'_κ, L'_κ and $TSys'_\kappa$ to denote the sets and the transition systems determined respectively by $(S_\kappa, \equiv_\kappa, I_\kappa)$ and $(S'_\kappa, \equiv'_\kappa, I'_\kappa)$. We shall construct a sequence of morphisms $(\bar{\sigma}_i, \lambda_i)\colon TSys_i \to TSys'_i$, which will determine a morphism $(\bar{\sigma}_\omega, \lambda_\omega)\colon TSys_\omega \to TSys'_\omega$, i.e., $dtsi\big((\sigma, \lambda)\big)$.

For $i \in \omega$, let σ_i be the function such that

$$\sigma_i(x) \quad = \quad \sigma(x) \quad \text{for } x \in S;$$

and

$$\sigma_i(xa) \quad = \quad \begin{cases} \sigma_i(x)\lambda_i(a) & \text{if } \lambda_i{\downarrow}a \\ \sigma_i(x) & \text{otherwise;} \end{cases}$$

where

$$\lambda_i(a) \quad = \quad \begin{cases} \lambda(a) & \text{if } \lambda(a) \notin \bigcup_{j<i} A'_j \\ \uparrow & \text{otherwise.} \end{cases}$$

Lemma 5.12

For all $i \in \omega$, we have that

i) $x \in S_i$ *implies* $\sigma_i(x) \in S'_i$;

ii) $x \equiv_i y$ *implies* $\sigma_i(x) \equiv'_i \sigma_i(y)$;

iii) $xa\ I_i\ yb$ *and* $\lambda_i{\downarrow}a$, $\lambda_i{\downarrow}b$ *implies* $\sigma_i(xa)\ I'_i\ \sigma_i(yb)$.

Proof. The three points are shown simultaneously by induction on i. The base case for $i = 0$ follows directly from the definition of σ_0 and from the fact that (σ, λ) is a morphism. Concerning the inductive step, the prove proceeds by first showing that points *(i)*, *(ii)* and *(iii)* hold for the generators of $(S_i, \equiv_i, I_i)$, and it concludes by checking that the closure rules preserve them. Both the tasks are fairly easy. ✓

It follows immediately from Lemma 5.12 that for $i \in \omega$, $\bar{\sigma}_i$, defined to be the map which sends $[x]_{\equiv_i}$ to $[\sigma_i(x)]_{\equiv'_i}$ is a well-defined function from $S_i/{\equiv_i}$ to $S'_i/{\equiv'_i}$. Then, the following lemma follows easily.

Lemma 5.13

For $i \in \omega$, the map $(\bar{\sigma}_i, \lambda_i)\colon TSys_i \to TSys'_i$ is a morphism of pre-transition systems with independence.

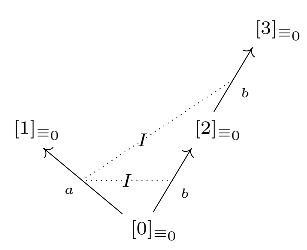
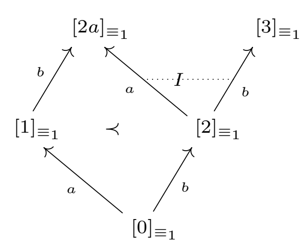
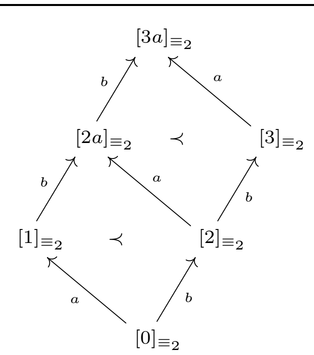
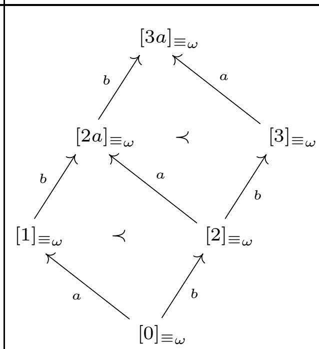
<table>
<tr><th></th><th>$TSys_\kappa$</th><th>$\equiv_\kappa$</th><th>I_κ</th></tr>
<tr><td>$\kappa = 0$</td><td>
</td><td>$[1]_{\equiv_0} = \{1, 0a\}$
$[2]_{\equiv_0} = \{2, 0b\}$
$[3]_{\equiv_0} = \{3, 2b, 0bb\}$</td><td>$[0a]_{\equiv_0}\ I_{\equiv_0}\ [0b]_{\equiv_0}$
$[0a]_{\equiv_0}\ I_{\equiv_0}\ [2b]_{\equiv_0}$</td></tr>
<tr><td>$\kappa = 1$</td><td>
</td><td>$[1]_{\equiv_1} = \{1, 0a\}$
$[2]_{\equiv_1} = \{2, 0b\}$
$[3]_{\equiv_1} = \{3, 2b, 0bb\}$
$[2a]_{\equiv_1} = \{2a, 1b, 0ab, 0ba\}$</td><td>$[0a]_{\equiv_1}\ I_{\equiv_1}\ [0b]_{\equiv_1}$
$[0a]_{\equiv_1}\ I_{\equiv_1}\ [0ab]_{\equiv_1}$
$[0b]_{\equiv_1}\ I_{\equiv_1}\ [0ba]_{\equiv_1}$
$[1b]_{\equiv_1}\ I_{\equiv_1}\ [2a]_{\equiv_1}$
$[0a]_{\equiv_1}\ I_{\equiv_1}\ [2b]_{\equiv_1}$
$[0ba]_{\equiv_1}\ I_{\equiv_1}\ [0bb]_{\equiv_1}$</td></tr>
<tr><td>$\kappa = 2$</td><td>
</td><td>$[0]_{\equiv_2} = \{0\}$
$[1]_{\equiv_2} = \{1, 0b\}$
$[2]_{\equiv_2} = \{2, 0a\}$
$[3]_{\equiv_2} = \{3, 2a, 0aa\}$
$[2b]_{\equiv_2} = \{1a, 2b, 0ba, 0ab\}$
$[3b]_{\equiv_2} = \{3b, 2ab, 2ba, 1aa,$
$0abb, 0baa, 0aba\}$</td><td>$[0a]_{\equiv_2}\ I_{\equiv_2}\ [0b]_{\equiv_2}$
$[0a]_{\equiv_2}\ I_{\equiv_2}\ [1b]_{\equiv_2}$
$[0b]_{\equiv_2}\ I_{\equiv_2}\ [2a]_{\equiv_2}$
$[1b]_{\equiv_2}\ I_{\equiv_2}\ [2a]_{\equiv_2}$
$[2a]_{\equiv_2}\ I_{\equiv_2}\ [2b]_{\equiv_2}$
$[2a]_{\equiv_2}\ I_{\equiv_2}\ [2bb]_{\equiv_2}$
$[2b]_{\equiv_2}\ I_{\equiv_2}\ [3a]_{\equiv_2}$
$[2bb]_{\equiv_2}\ I_{\equiv_2}\ [3a]_{\equiv_2}$</td></tr>
<tr><td>$\kappa = \omega$</td><td>
</td><td colspan="2"><u>**Comments**</u>. Suppose that the construction starts from $TSys_0$, where the dotted lines indicate relation I. $TSys_0$ fails to a be transition system with independence because there is no diamond for the transitions sticking out $[0]_{\equiv_0}$. In $TSys_1$ this problem has been solved by use of *(D1)*. However, now there is no diamond for the transitions leaving from $[2]_{\equiv_1}$, which are independent because of the closure *(Cl3)*. The problem is fixed in $TSys_2$ which is a transition system with independence and coincides with $TSys_\omega$.</td></tr>
</table>

Figure 3: An example of the construction of $TSys_\omega$.

For any $i \in \omega$, consider the morphism of pre-transition systems with independence $(in_i'^{\omega}, id_i'^{\omega}) \circ (\bar{\sigma}_i, \lambda_i)\colon TSys_i \to TSys'_{\omega}$. Recall that for $x \in S_i$, we have that $\sigma_{i+1}(x \restriction A_i) = \sigma_i(x \restriction A_i) \restriction A'_i = \sigma_i(x) \restriction A'_i$, from which it follows that $\sigma_{i+1}(x \restriction A_i) \restriction A'_{\omega} = \sigma_i(x) \restriction A'_{\omega}$. Then

$$\begin{aligned} in_i'^{\omega} \circ \bar{\sigma}_i\big([x]_{\equiv_i}\big) &= in_i'^{\omega}\big([\sigma_i(x)]_{\equiv'_i}\big) = \big[\sigma_i(x) \restriction A'_{\omega}\big]_{\equiv'_{\omega}} \\ &= \big[\sigma_{i+1}(x \restriction A_i) \restriction A'_{\omega}\big]_{\equiv'_{\omega}} = in_{i+1}'^{\omega}\big([\sigma_{i+1}(x \restriction A_i)]_{\equiv'_{i+1}}\big) \\ &= in_{i+1}'^{\omega} \circ \bar{\sigma}_{i+1}\big([x \restriction A_i]_{\equiv_{i+1}}\big) = in_{i+1}'^{\omega} \circ \bar{\sigma}_{i+1} \circ in_{i+1}\big([x]_{\equiv_i}\big), \end{aligned}$$

i.e., $in_i'^{\omega} \circ \bar{\sigma}_i = in_{i+1}'^{\omega} \circ \bar{\sigma}_{i+1} \circ in_{i+1}$ for any $i \in \omega$. Moreover, since $a \in A_i$ implies $\lambda(a) \in A'_i$, it is easy to see that $id_i'^{\omega} \circ \lambda_i = id_{i+1}'^{\omega} \circ \lambda_{i+1} \circ id_{i+1}$ for any $i \in \omega$. Thus, we have that

$$\big\{ (in_i'^{\omega}, id_i'^{\omega}) \circ (\bar{\sigma}_i, \lambda_i)\colon TSys_i \to TSys'_{\omega} \;\big|\; i \in \omega \big\}$$

is a cocone for the $\boldsymbol{\omega}$-diagram $\mathcal{D}$ given in Proposition 5.10. Then, there exists a unique $(\bar{\sigma}_{\omega}, \lambda_{\omega})\colon TSys_{\omega} \to TSys'_{\omega}$ induced by the colimit construction, which is the morphism of transition systems with independence we associate to (σ, λ), i.e., $dtsi\big((\sigma, \lambda)\big) = (\bar{\sigma}_{\omega}, \lambda_{\omega})$. From Proposition 5.10, it is immediate to see that $\bar{\sigma}_{\omega}([x]_{\equiv_{\omega}}) = [\sigma_i(\bar{x}) \restriction A'_{\omega}]_{\equiv'_{\omega}}$ for $\bar{x} \in S_i$ such that $\bar{x} \restriction A_{\omega} = x$, or, equivalently, $\bar{\sigma}_{\omega}([x]_{\equiv_{\omega}}) = [\sigma_i(x) \restriction A'_{\omega}]_{\equiv'_{\omega}}$ for any i such that $x \in S_i$, and that

$$\lambda_{\omega}(a) = \begin{cases} \lambda(a) & \text{if } \lambda(a) \notin A'_{\omega} \\ \uparrow & \text{otherwise.} \end{cases}$$

The following proposition follows directly from the universal properties of colimits.

Proposition 5.14 (*dtsi*: $\underline{\mathsf{TSI}} \to \underline{\mathsf{dTSI}}$ *is a functor*)
The map dtsi is a functor from $\underline{\mathsf{TSI}}$ *to* $\underline{\mathsf{dTSI}}$.

The question we address next concerns what we get when we apply *dtsi* to a deterministic transition system with independence DTI. We shall see that in this case the inductive construction of $TSys_{\omega}$ gives a transition system which is isomorphic to DTI. More precisely, each $\equiv_{\omega}$-equivalence class of $(S_{DTI})_{\omega}$ contains exactly *one* state of the original transition system, and the transition system with independence morphism $(in_0^{\omega} \circ in, id_0^{\omega})\colon DTI \to dtsi(DTI)$—whose transition component sends $s \in S_{DTI}$ to $[s]_{\equiv_{\omega}}$—is actually an isomorphism. Moreover, we shall see that its inverse (ε, id), where $\varepsilon([x]_{\equiv_{\omega}})$ is the unique $s \in S_{DTI}$ such that $s \equiv_{\omega} x$, is the counit of the adjunction.

Lemma 5.15
Let $DTI = (S, s^I, L, Tran, I)$ *be a deterministic transition system with independence. Then,* $(S_1, \equiv_1, I_1)$ *coincides with* $(S_0, \equiv_0, I_0)$*. Therefore,* $(in_0^{\omega} \circ in, id_0^{\omega})$ *is an isomorphism whose inverse is* (ε, id).

Proof. We already know from Proposition 5.8 that (in, id) is an isomorphism if DTI is deterministic. Thus, $(in_0^\omega \circ in, id_0^\omega)$ is an isomorphism if and only if $(in_0^\omega, id_0^\omega)\colon TSys_0 \to TSys_\omega$ is so, which, in turn, is a consequence of the first part of the claim.

Observe that $A_0 = \varnothing$ and, therefore, $TA_0 = \varnothing$. In fact, since DTI and $TSys_0$ are isomorphic, if there were $xa\ I_0\ xa$, then I_{DTI} would not be irreflexive. Then, in order to show that $(S_1, \equiv_1, I_1) = (S_0, \equiv_0, I_0)$, it is enough to see that no new elements are introduced by *(D1)* and *(D2)*. In fact, in this case, $(S_1, \equiv_1, I_1)$ would be the least *suitable* triple which contains $(S_0, \equiv_0, I_0)$ which is clearly $(S_0, \equiv_0, I_0)$ itself.

(D1) Suppose $xa\ I_0\ xb$. Then, by Corollary 5.6, there exist $s,\ s',\ s'' \in S$ such that $s \equiv_0 x$, $s' \equiv_0 xa$ and $s'' \equiv_0 xb$. Therefore, by Lemma 5.5, we have $(s, a, s')\ I\ (s, b, s'')$ in *Tran*. Since DTI is a transition system with independence, there exists u such that $Diam_{a,b}(s, s', s'', u)$, and then we have $sab \equiv_0 u \equiv_0 sba$ and, therefore, by *(Cl1)*, we already have $xab \equiv_0 xba$ in $(S_0, \equiv_0, I_0)$.

(D2) Analogous to the previous case. ✓

Thus, we have proved the following corollary.

Corollary 5.16
$(\varepsilon, id)\colon dtsi(DTI) \to DTI$ *is a transition system with independence isomorphism.*

Before showing that (ε, id) is the counit of the reflection of dTSI in TSI, we need the following lemma which characterises the behaviour of transition system with independence morphisms whose target is deterministic.

Lemma 5.17
Let DTI be a deterministic transition system with independence and consider a morphism $(\sigma, \lambda)\colon TI \to DTI$ in TSI. Let $TSys_\kappa$, $\kappa \in \omega \cup \{\omega\}$ be the sequence of pre-transition systems with independence associated to TI. Consider $a \in L_{TI}$ and suppose that $a \in A_i$. Then $\lambda\!\uparrow\! a$.

Proof. Consider the sequence of pre-transition systems with independence $TSys'_\kappa$ associated to DTI and the morphisms $(\bar{\sigma}_i, \lambda_i)\colon TSys_i \to TSys'_i$. Since, as it follows from Lemma 5.15, $TSys'_i = TSys'_\omega$ for any $i \in \omega$, the morphisms $(\bar{\sigma}_i, \lambda_i)\colon TSys_i \to TSys'_\omega$ form a cocone for the ω-diagram which defines $TSys_\omega$. Moreover, we have that any λ_i coincides with λ, because $A'_i = \varnothing$. Then, if $a \in A_i$, reasoning as in the proof of Proposition 5.10, we have that $\lambda_j\!\uparrow\! a$ for any $j \le i$, i.e., $\lambda\!\uparrow\! a$. ✓

We are ready now to show that (ε, id) is couniversal.

Proposition 5.18 ($(\varepsilon, id)\colon dtsi(DTI) \to DTI$ *is couniversal*)
For any transition system with independence TI, deterministic transition system with independence DTI, and morphism $(\varphi, \mu)\colon dtsi(TI) \to DTI$, there exists a unique $k\colon TI \to DTI$ such that $(\varepsilon, id) \circ dtsi(k) = (\varphi, \mu)$.

$$
\begin{tikzcd}
dtsi(DTI) \arrow[r, "(\varepsilon, id)"] & DTI \\
dtsi(TI) \arrow[u, "dtsi(k)"] \arrow[ur, "(\varphi, \mu)"'] &
\end{tikzcd}
$$

Proof. Let us consider $k = (\sigma, \lambda)$, where $\sigma(s) = \varphi([s]_{\equiv_\omega})$ and λ is the function which coincides with μ on $(L_{TI})_\omega$ and is undefined elsewhere. Observe that this is the only possible choice for k. In fact, any $k' \colon TI \to DTI$ which has to make the diagram commute must be of the kind (σ', λ') with $\lambda'(a) = \mu(a) = \lambda(a)$ for $a \in (L_{TI})_\omega$. Moreover, by Lemma 5.17, if $a \in A_\omega$, it must be $\lambda' {\uparrow} a$, i.e., $\lambda' = \lambda$. Furthermore, $\sigma'(s)$ must be an $\bar{s}$ in S_{DTI} such that $\varepsilon([\bar{s}]_{\equiv_\omega}) = \bar{s}$ coincides with $\varphi([s]_{\equiv_\omega})$, i.e., σ' is the σ we have chosen.

In order to show that (σ, λ) is a morphism of pre-transition systems with independence, it is enough to observe that (σ, λ) can be expressed as the composition of the morphisms of transition systems with independence $(\varphi, \mu) \circ (in_0^\omega \circ in, id_0^\omega) \colon TI \to dtsi(TI) \to DTI$. This makes easy to conclude the proof. ✓

THEOREM 5.19 ($dtsi \dashv \hookleftarrow$)
Functor $dtsi$ is left adjoint to the inclusion functor $\underline{\mathsf{dTSI}} \hookrightarrow \underline{\mathsf{TSI}}$. Therefore, the adjunction $\langle dtsi, \hookleftarrow \rangle \colon \underline{\mathsf{dTSI}} \rightharpoonup \underline{\mathsf{TSI}}$ is a reflection.

The adjunction $\underline{\mathsf{dTSI}} \hookrightarrow\!\!\!\lhd\, \underline{\mathsf{TSI}}$ that we have so established closes another face of the cube. In particular, we have obtained the following square, which matches the one presented in Section 2.

$$\begin{tikzcd}
\underline{\mathsf{TSI}} & \underline{\mathsf{TS}} \arrow[l, hook', -{Triangle[open]}] \\
\underline{\mathsf{dTSI}} \arrow[u, hook, -{Triangle[open]}] & \underline{\mathsf{dTS}} \arrow[l, hook', -{Triangle[open]}] \arrow[u, hook, -{Triangle[open]}]
\end{tikzcd}$$

6 Deterministic Labelled Event Structures

In this section we prove that there exists a reflection from the category of deterministic labelled event structures to labelled event structures. A reflection $\underline{\mathsf{dLES}} \hookrightarrow\!\!\!\lhd\, \underline{\mathsf{LES}}$ does exist, for it follows from the reflections we have presented in the previous sections. In fact, the results in Section 4 and 5 show that there exist adjunctions

$$\underline{\mathsf{dLES}} \hookrightarrow\!\!\!\rhd\, \underline{\mathsf{dTSI}} \hookrightarrow\!\!\!\lhd\, \underline{\mathsf{TSI}} \lhd\!\!\!\hookleftarrow \underline{\mathsf{LES}}.$$

Now, in order to show that there is a coreflection from $\underline{\mathsf{dLES}}$ to $\underline{\mathsf{LES}}$, since $\underline{\mathsf{dLES}} \cong \underline{\mathsf{doTSI}}_{\mathsf{E}}$ and $\underline{\mathsf{LES}} \cong \underline{\mathsf{oTSI}}_{\mathsf{E}}$, it is enough to show that $\underline{\mathsf{dTSI}} \hookrightarrow\!\!\!\lhd\, \underline{\mathsf{TSI}}$ cuts down to a reflection $\underline{\mathsf{doTSI}}_{\mathsf{E}} \hookrightarrow\!\!\!\lhd\, \underline{\mathsf{oTSI}}_{\mathsf{E}}$. In this case, we have an adjunction

$$\underline{\mathsf{dLES}} \cong \underline{\mathsf{doTSI}}_{\mathsf{E}} \hookrightarrow\!\!\!\lhd\, \underline{\mathsf{oTSI}}_{\mathsf{E}} \cong \underline{\mathsf{LES}},$$

whose right adjoint is isomorphic to the inclusion functor $\underline{\mathsf{dLES}} \hookrightarrow \underline{\mathsf{LES}}$. Intuitively, the left adjoint $dles \colon \underline{\mathsf{LES}} \to \underline{\mathsf{dLES}}$ is obtained by considering the occurrence transition system with independence $les.otsi(ES)$ of the finite configurations of ES, constructing its deterministic version by applying $dtsi$, and then considering the labelled event structure associated with such a deterministic transition system with independence, by means of *otsi.les*.

As usual, to establish that $\underline{\mathsf{dTSI}} \hookrightarrow\!\!\lhd \underline{\mathsf{TSI}}$ restricts to $\underline{\mathsf{doTSI}}_{\mathsf{E}} \hookrightarrow\!\!\lhd \underline{\mathsf{oTSI}}_{\mathsf{E}}$, it is enough to show that if OTI is an occurrence transition system with independence, then so is $dtsi(OTI)$, and that $dtsi(OTI)$ satisfies (E) whenever OTI does. Of course, this also proves that $\underline{\mathsf{oTSI}} \hookrightarrow\!\!\lhd \underline{\mathsf{TS}}$ restricts to $\underline{\mathsf{doTSI}} \hookrightarrow\!\!\lhd \underline{\mathsf{oTSI}}$.

In the following, let OTI be an occurrence transition system with independence and let $(S_\kappa, \equiv_\kappa, I_\kappa)$ and $TSys_\kappa$, $\kappa \in \omega \cup \{\omega\}$, be the sequences of suitable triples and pre-transition systems with independence which define $dtsi(OTI)$.

PROPOSITION 6.1 ($\underline{\mathsf{doTSI}} \hookrightarrow\!\!\lhd \underline{\mathsf{oTSI}}$)
If OTI is an occurrence transition system with independence, so is $dtsi(OTI)$.

Proof. Recall from Section 4 that the states of OTI is equipped with a 'depth', namely the length of the paths leading to it. Moreover, there is a transition $s \xrightarrow{a} s'$ only if $depth(s') = depth(s) + 1$.

Observe now that $TSys_0$ is reachable and acyclic. To this purpose, recall that (the transition system underlying) $TSys_0$ is obtained from (the transition system underlying) OTI modulo the least equivalence which identifies states reachable from a common state by two equally-labelled sequences of transitions. Since OTI is reachable, this reduces to say that $s \equiv_0 s'$ if and only if there are paths π_s and $\pi_{s'}$ in OTI such that $Act(\pi_s) = Act(\pi_{s'})$, which implies that $depth(s) = depth(s')$ whenever $s \equiv_0 s'$. This makes our claims obvious, showing also that all the paths in $TSys_0$ leading to the same state have the same length, i.e., that *depth* extends smoothly to the states of $TSys_0$.

A direct inspection of the closure properties *(Cl1)*, *(Cl2)* and *(Cl3)*, of the rules *($\Im$)*, *(D1)* and *(D2)*, and of the definition of $TSys_\omega$ shows that all the $TSys_\kappa$, and in particular $dtsi(OTI) = TSys_\omega$, are reachable, acyclic and have a notion of 'depth' defined by the length of their paths.

Concerning the property of occurrence transition systems with independence, we prove by induction on $depth\big([z]_{\equiv_\omega}\big)$ that, if $\big([y']_{\equiv_\omega}, b, [z]_{\equiv_\omega}\big)$ and $\big([y'']_{\equiv_\omega}, a, [z]_{\equiv_\omega}\big)$ are *distinct* transitions of $TSys_\omega$, then there exists a state $[x]_{\equiv_\omega}$ in $TSys_\omega$ such that $Diam_{a,b}\big([x]_{\equiv_\omega}, [y']_{\equiv_\omega}, [y'']_{\equiv_\omega}, [z]_{\equiv_\omega}\big)$

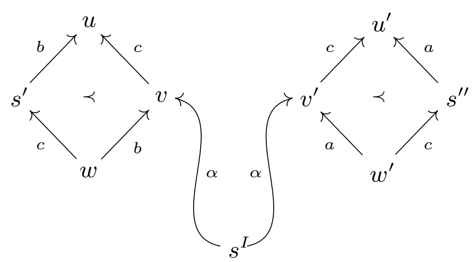
($depth \leq 1$). Vacuous, since $dtsi(OTI)$ is reachable and acyclic.

($depth > 1$). It is enough to show that if $\big([y']_{\equiv_i}, b, [z]_{\equiv_i}\big)$ and $\big([y'']_{\equiv_i}, a, [z]_{\equiv_i}\big)$ belong to $TSys_i$, i.e., $y'b \equiv_i z \equiv_i y''a$, then the required diamond exists in $TSys_\omega$. We proceed by induction on i.

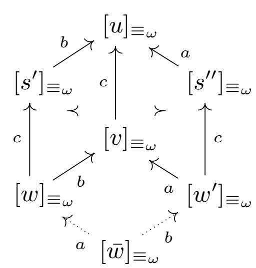
($i = 0$). Since both transitions belong to $TSys_0$, there are (s', b, u) and (s'', a, u') in OTI such that $s' \equiv_\omega y'$, $s'' \equiv_\omega y''$, $u \equiv_\omega z \equiv_\omega u'$, and $u \equiv_0 u'$. Observe that, due to the possible collapsing of autoindependent transitions, there can be more that one pair of such transitions. Without loss of generality, we can assume u and u' chosen at minimal depth in OTI.

By definition, since $u \equiv_0 u'$ there exist paths π_u and $\pi_{u'}$ in OTI such that $Act(\pi_u) = \alpha c = Act(\pi_{u'})$. Let (v, c, u) and (v', c, u') be the last transitions on these paths. Since v and v' are reachable via α-labelled paths, we have $v \equiv_0 v'$. Observe that $c \notin A_\omega$. In fact, if $c \in A_\omega$, since $a, b \notin A_\omega$, it would be $(v, c, u) \neq (s', b, u)$ and $(v', c, u') \neq (s'', a, u')$. Then, by the property of occurrence transition systems with independence, there would be w and w' in OTI such that $Diam_{c,b}(w, s', v, u)$ and $Diam_{c,a}(w', s'', v', u')$ and, therefore, (w, b, v) and (w', a, v') with $w \equiv_\omega y'$, $w' \equiv_\omega y''$, $v \equiv_\omega z \equiv_\omega v'$, and $depth(v) < depth(u)$, contradicting our assumption.

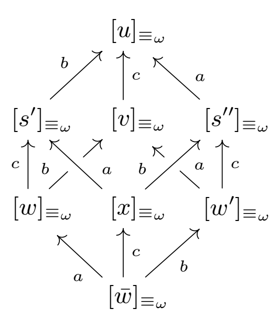
Since $([v]_{\equiv_0}, c, [u]_{\equiv_0}) = ([v']_{\equiv_0}, c, [u']_{\equiv_0})$, it follows that, if $(v, c, u) = (s', b, u)$ and $(v', c, u') = (s'', a, u')$, then $([s']_{\equiv_\omega}, b, [u]_{\equiv_\omega}) = ([s'']_{\equiv_\omega}, a, [u']_{\equiv_\omega})$, and there is nothing to show. Therefore, without loss of generality, assume $(v, c, u) \neq (s', b, u)$. Then there exists w in OTI such that $Diam_{c,b}(w, s', v, u)$. In case, $(v', c, u') = (s'', a, u')$, we have $([s'']_{\equiv_\omega}, a, [u']_{\equiv_\omega}) = ([v]_{\equiv_\omega}, c, [u]_{\equiv_\omega})$ and, therefore, the required diamond $Diam_{a,b}\big([w]_{\equiv_\omega}, [s']_{\equiv_\omega}, [s'']_{\equiv_\omega}, [u]_{\equiv_\omega}\big)$. Finally, if instead $(v', c, u') \neq (s'', a, u')$, there exists w' in OTI such that $Diam_{c,a}(w', s'', v', u')$. The situation is illustrated by the following picture.



Since $v \equiv_0 v'$, the transitions $([w]_{\equiv_0}, b, [v]_{\equiv_0})$ and $([w']_{\equiv_0}, a, [v]_{\equiv_0})$ belong to $TSys_0$. We can assume that these are distinct, since $[w]_{\equiv_0} = [w']_{\equiv_0}$ and $a = b$ implies again that $([s']_{\equiv_0}, b, [u]_{\equiv_0}) = ([s'']_{\equiv_0}, a, [u]_{\equiv_0})$. Then, since $c \notin A_\omega$ and, therefore, $depth([v]_{\equiv_\omega}) < depth([z]_{\equiv_\omega})$ in $TSys_\omega$, by induction hypothesis, there exists $[\bar{w}]_{\equiv_\omega}$ such that $Diam_{a,b}\big([\bar{w}]_{\equiv_\omega}, [w]_{\equiv_\omega}, [w']_{\equiv_\omega}, [v]_{\equiv_\omega}\big)$. Therefore, we have the following situation in $TSys_\omega$.



Then, since $TSys_\omega$ is a transition system with independence, by properties *(i)*, *(iii)* and *(iv)* in Definition 3.7, there exists $[x]_{\equiv_\omega}$ completing the diagram to a cube as in the picture below.



Clearly, it is $Diam_{a,b}\big([x]_{\equiv_\omega}, [s']_{\equiv_\omega}, [s'']_{\equiv_\omega}, [u]_{\equiv_\omega}\big)$, concluding this part of the proof.

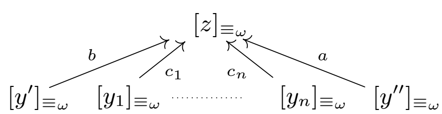
$(i > 0)$. We proceed by case analysis inspecting the rules that generated $y'b \equiv_i y''a$. We start by proving the thesis for the generators ${}_\gamma S_i$, ${}_\gamma\!\equiv_i$, and ${}_\gamma I_i$ of $(S_i, \equiv_i, I_i)$.

($\Im$). Then we have $\bar{y}'b \equiv_{i-1} \bar{y}''a$, for some $\bar{y}'$ and $\bar{y}''$ such that $\bar{y}' \restriction A_{i-1} = y'$ and $\bar{y}'' \restriction A_{i-1} = y''$, and the thesis follows by (inner) induction.

(D1) & (D2). Then $y'b \equiv_i y''a$ arises from completing either $xa\ I_{i-1}\ xb$, with $xa \equiv_{i-1} y'$ and $xb \equiv_{i-1} y''$, or $xa\ I_{i-1}\ xab$, with $xa \equiv_{i-1} y'$ and $xb \equiv_i y''$, or, symmetrically, $xb\ I_{i-1}\ xba$, with $xb \equiv_{i-1} y''$ and $xa \equiv_i y'$. In all cases, we have $Diam_{a,b}\big([x]_{\equiv_i}, [y']_{\equiv_i}, [y'']_{\equiv_i}, [z]_{\equiv_i}\big)$ in $TSys_i$ and therefore, since $a, b \not\in A_\omega$, the required diamond in $TSys_\omega$.

Concerning the closure properties, observe that *(Cl2)* and *(Cl3)* do not alter $\equiv_i$. If instead $y'b \equiv_i y''a$ follows from *(Cl1)*, we have $y' \equiv_i y''$ and $a = b$, which means that $\big([y']_{\equiv_\omega}, b, [z]_{\equiv_\omega}\big) = \big([y'']_{\equiv_\omega}, a, [z]_{\equiv_\omega}\big)$. Therefore, in order to conclude the proof, we only need to analyse the case in which $y'b \equiv_i y''a$ is induced by closing transitively ${}_\gamma\!\equiv_i$, i.e., when $y'b\ {}_\gamma\!\equiv_i\ y_1c_1\ {}_\gamma\!\equiv_i \cdots {}_\gamma\!\equiv_i\ y_nc_n\ {}_\gamma\!\equiv_i\ y''a$. We proceed by induction on n, the base case being already proved.

(induction step). The situation in $TSys_\omega$ is illustrated by the following figure.



By the previous part of this proof, there exists $[w]_{\equiv_\omega}$ such that, in $TSys_\omega$, we have $Diam_{b,c_1}\big([w]_{\equiv_i}, [y']_{\equiv_i}, [y_1]_{\equiv_i}, [z]_{\equiv_i}\big)$, and, by induction on n, there is $[w']_{\equiv_\omega}$ such that $Diam_{c_1,a}\big([w']_{\equiv_i}, [y_1]_{\equiv_i}, [y'']_{\equiv_i}, [z]_{\equiv_i}\big)$. Since $depth([y_1]_{\equiv_\omega}) < depth([z]_{\equiv_\omega})$ in $TSys_\omega$, we are in the condition of exploiting the (outer) induction hypothesis and concluding the proof as for the case $(i = 0)$. ✓

PROPOSITION 6.2 ($\underline{\mathsf{doTSI}}_{\mathsf{E}} \hookrightarrow\!\!\lhd\ \underline{\mathsf{oTSI}}_{\mathsf{E}}$)
If OTI satisfies (E), then $dtsi(OTI)$ satisfies (E).

Proof. Observe that $TSys_0$ clearly enjoys (E), and that (E) is preserved by the rules *($\Im$)*, *(D1)* and *(D2)* and by the closures *(Cl1)*, *(Cl2)* and *(Cl3)*. ✓

Therefore, defining $dles\colon \underline{\mathsf{LES}} \to \underline{\mathsf{dLES}}$ as $otsi.les \circ dtsi \circ les.otsi$ we have the following result.

THEOREM 6.3 ($dles \dashv \hookrightarrow$)
The mapping dles extends to a functor which is left adjoint of the inclusion of $\underline{\mathsf{dLES}}$ in $\underline{\mathsf{LES}}$. Then, $\langle dles, \hookrightarrow\rangle$ is a reflection.

An example of the construction is given in Figure 4.

The coreflection $\underline{\mathsf{dLES}} \hookrightarrow\!\!\lhd\ \underline{\mathsf{LES}}$ closes the last two faces of the cube. So, our results may be summed up in the following cube of relationships among models.

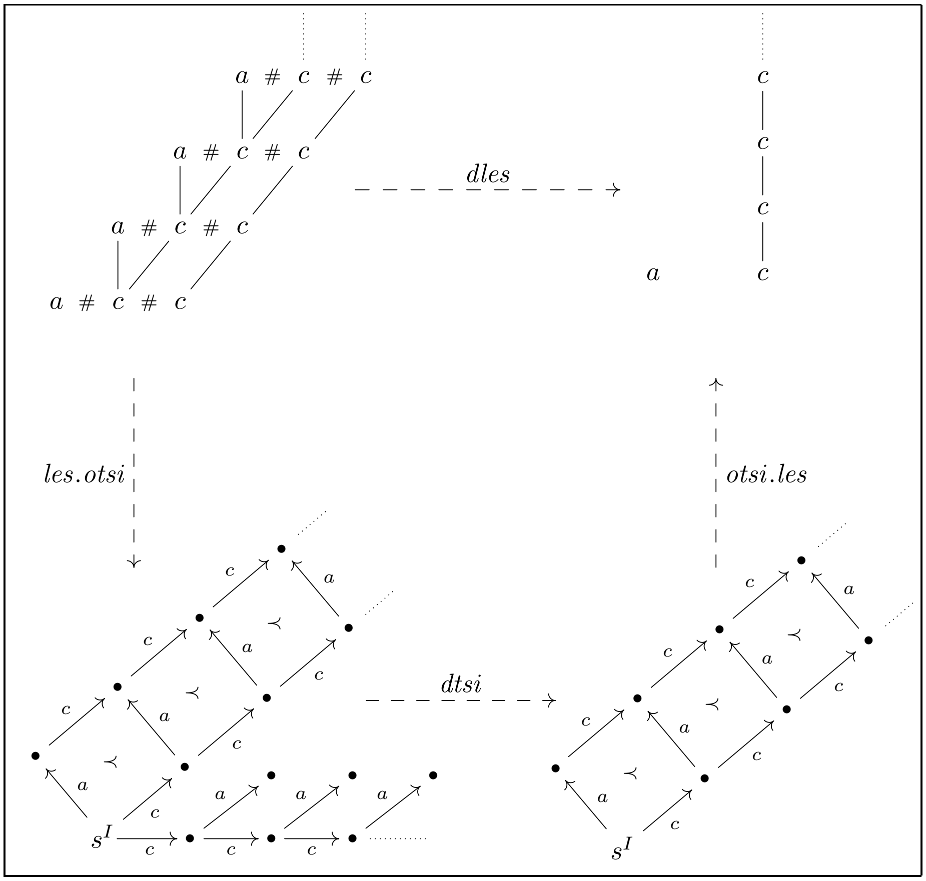
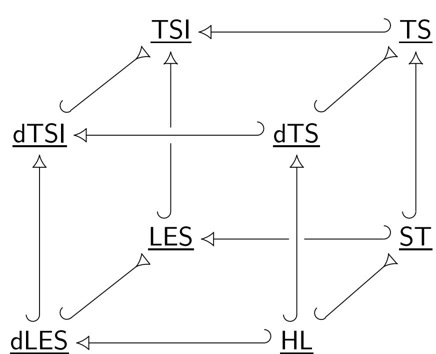


Figure 4: An event structure *ES* and *dles*(*ES*)

Theorem 6.4 (*The Cube*)



An alternative construction for *dles*

It may be interesting to notice that, since $TSys_i$ is not a transition system with independence, the sequence $\{TSys_i\}_{i\in\omega}$ which defines $dtsi(les.otsi(ES))$ does not correspond to a sequence of labelled event structures. Nevertheless, a sequence $\{Ev_i\}_{i\in\omega}$ which characterises $dles(ES)$ as a colimit in $\underline{\mathsf{LES}}$ exists. In the following, we shall report only the relevant definitions, omitting all the proofs, which can be found in [11, 14].

Similarly to Section 5, we shall proceed by defining a sequence of triples $(\sim_i, \leq_i, \#_i)$, each representing a *quotient* of the original labelled event structure in which—informally speaking—the *'degree'* of nondeterminism has decreased. The colimit of such a sequence will represent a deterministic event structure isomorphic to $dles(ES)$. Also in this case, the only way to cope with *autoconcurrency* is by eliminating it. However, the reader will notice that the task is now much easier than in the case of transition systems with independence.

Let $ES = (E, \#, \leq, \ell, L)$ be a labelled event structure, $A(ES)$ denote the 'autoconcurrency' set $\{a \in L \mid \exists e, e' \in E, e \ co\ e' \text{ and } \ell(e) = a = \ell(e')\}$ and $NA(ES) = \{e \in E \mid \ell(e) \notin A(ES)\}$ the associated set of 'non-autoconcurrent' events. Consider the sequence of relations $(\sim_\kappa, \leq_\kappa, \#_\kappa)$, for $\kappa \in \omega \cup \{\omega\}$, where

- $\sim_0 = \big\{(e,e) \bigm| e \in NA(ES)\big\}; \quad \leq_0 = \leq; \quad \#_0 = \#;$

for $i > 0$,

- $\sim_i$ is the least equivalence on $NA(ES)$ such that

  i) $\sim_{i-1} \subseteq \sim_i$;

  ii)
$$\begin{aligned} & e \not\leq_{i-1} e',\ e' \not\leq_{i-1} e,\ \ell(e) = \ell(e') \\ & \qquad \lfloor e\rfloor_{\leq_{i-1}} \not\#_{i-1} \lfloor e'\rfloor_{\leq_{i-1}} \setminus \{e'\} \quad \text{and} \\ & \qquad\qquad\qquad \lfloor e\rfloor_{\leq_{i-1}} \not\#_{i-1} \lfloor e'\rfloor_{\leq_{i-1}} \setminus \{e'\} \\ & \qquad\qquad\qquad\qquad\qquad \text{implies } e \sim_i e', \end{aligned}$$

  where $\lfloor e\rfloor_{\leq_i}$ stands for $\{e' \in NA(ES) \mid e' \leq_i e\}$ and, for $x, y \subseteq NA(ES)$, $x \not\#_i y$ is a shorthand for $\forall\, e \in x,\ \forall\, e' \in y,\ e \not\#_i e'$.

- $e \leq_i e'$ if and only if $\forall\, \bar{e}' \sim_i e'\ \exists\, \bar{e} \sim_i e.\ \bar{e} \leq_{i-1} \bar{e}'$;

- $e \#_i e'$ if and only if $\forall \bar{e}' \sim_i e'\ \forall \bar{e} \sim_i e.\ e \#_{i-1} e'$;

and finally, for $\kappa = \omega$,

$$\sim_\omega = \bigcup_{i\in\omega} \sim_i, \qquad \leq_\omega = \bigcup_{i\in\omega} \bigcap_{j>i} \leq_i, \qquad \#_\omega = \bigcap_{i\in\omega} \#_i$$

Then, for $k \in \omega \cup \{\omega\}$, define

$$Ev_\kappa = (NA(ES)/\sim_\kappa, \leq_{\sim_\kappa}, \#_{\sim_\kappa}, \ell_{\sim_\kappa}, L \setminus A(ES)),$$

where

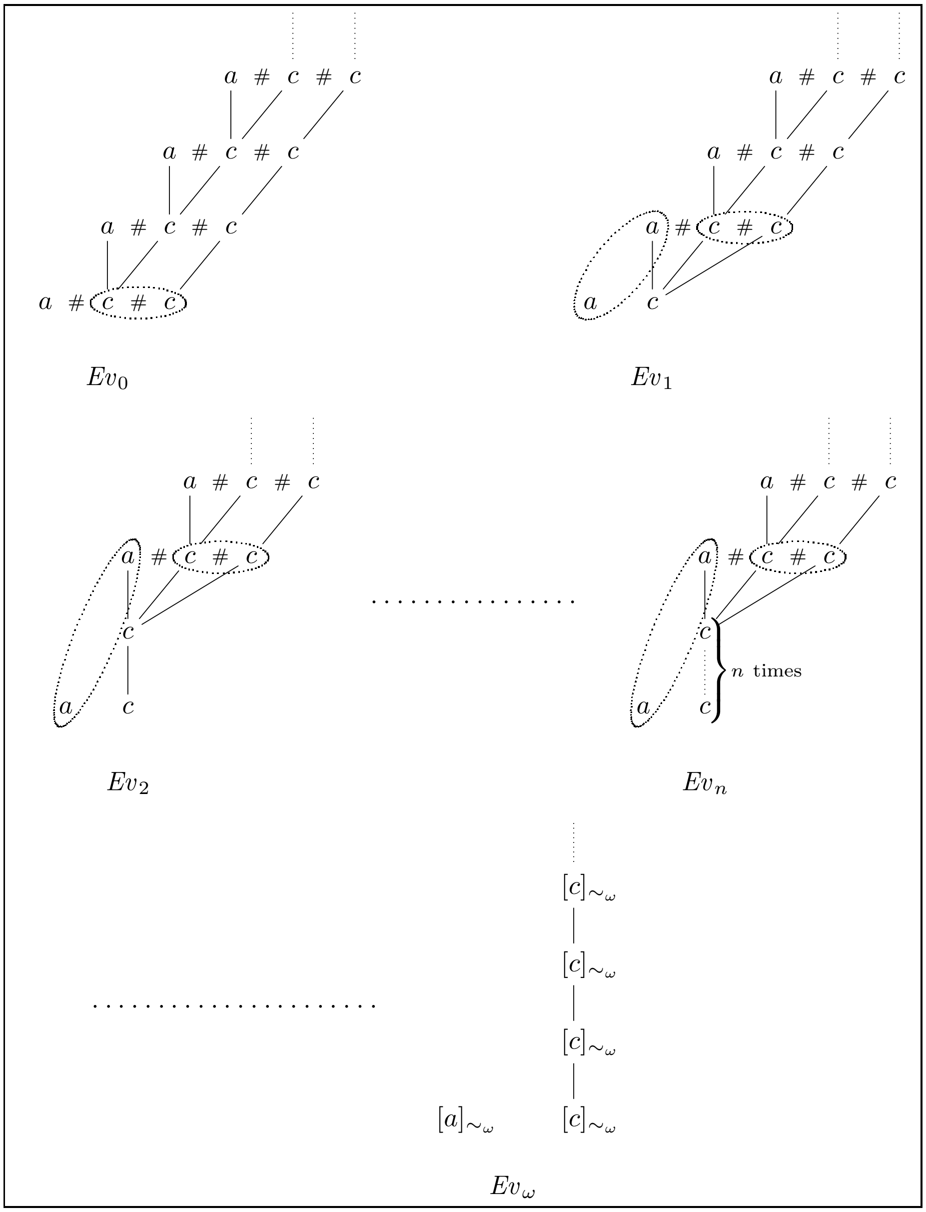


Figure 5: The alternative construction of $dles(ES)$ for ES in Figure 4.

- $NA(ES)/\sim_\kappa$ is the set of $\sim_\kappa$-classes of $NA(ES)$,
- $[e]_{\sim_\kappa} \leq_{\sim_\kappa} [e']_{\sim_\kappa}$ if and only if $e \leq_\kappa e'$,
- $[e]_{\sim_\kappa} \#_{\sim_\kappa} [e']_{\sim_\kappa}$ if and only if $e \#_\kappa e'$,
- $\ell_{\sim_\kappa}([e]_{\sim_\kappa}) = \ell(e)$.

It is proved in [11, 14] that the mapping $ES \mapsto Ev_\omega$ is (the object component of) a left adjoint to the inclusion $\underline{\mathsf{dLES}} \hookrightarrow \underline{\mathsf{LES}}$. It follows that Ev_ω is isomorphic to $dles(ES)$.

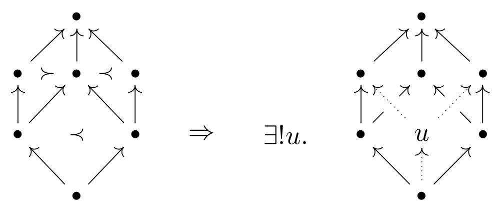
Figure 5 shows the sequence $\{Ev_\kappa\}_\kappa$ for the labelled event structure of Figure 4. The dotted ovals in Ev_i represent the events collapsed by $\sim_{i+1}$. In Ev_ω, the classes $[a]_{\sim_\omega}$ and $[c]_{\sim_\omega}$ at level i contain, respectively, all the a-labelled events and the two c-labelled events at level i of the original event structure.

Conclusion

We have established a complete 'cube' of formal relationships between well-known models for concurrency (and a new one). Thus, we have a complete picture of how to translate between these models via adjunctions along the axes of 'interleaving/noninterleaving', 'linear/branching' and 'behaviour/system'. Notice also the pleasant conformity in the picture, with *coreflections* along the 'interleaving/noninterleaving' and 'behaviour/system' axes, and *reflections* along 'linear/branching'.

A relevant role in this paper is played by the *occurrence transition systems with independence*, which turn out to be a slight generalisation of labelled event structures and, therefore, to allow an easy, interesting characterisation of coherent, finitary, prime algebraic domains. Concerning transition systems with independence, it is worth remarking that $\underline{\mathsf{TSI}}$ embeds fully and faithfully in the category of asynchronous transition systems via an easy construction: given TI, considering its underlying transition system, label each transition with its $\sim$-equivalence class, and take the independence inherited by TI. Unfortunately, about the relationships between asynchronous transition systems and transition systems with independence it does not seem possible to give more than this embedding, since it, together with other natural ones, fails to enjoy any universal property.

Axiom *(i)* of transition systems with independence, depending non trivially on $\sim$, represents a *'global'* constraint, as opposed to the others, which involve only local information. This may be considered a slightly unpleasant feature of our definition. It is an open question whether there exists alternative axiomatics for transition systems with independence. However, one can identify *weaker* sets of axioms and yield kinds of 'generalised' transition systems with independence which still enjoy important properties. For instance, removing axiom *(i)*, replacing 'there exists…' by 'there exists a unique…' in *(ii)* and *(iii)*, and adding the following axiom



one obtains a category strictly larger than $\underline{\mathsf{TSI}}$ which can replace it in the cube. It may be interesting to remark that the axioms above, together with the conditions of Definition 4.1, define exactly *occurrence* transition systems with independence.

It is worth remarking here that all the adjunctions in this paper would still hold if we modified uniformly the morphisms of the involved categories by allowing label components which, where defined, act identically. However, if we considered only total morphisms, the reflections $\underline{\mathsf{dTSI}} \hookrightarrow\!\!\!\lhd\ \underline{\mathsf{TSI}}$ and $\underline{\mathsf{dLES}} \hookrightarrow\!\!\!\lhd\ \underline{\mathsf{LES}}$ would not exist.

Although the choice of deterministic labelled event structures for behavioural, linear and noninterleaving models is sensible, it is not the unique possible choice. For instance, in [16] the authors introduce a category of *pomset languages* and a category of *generalised trace languages* which can replace $\underline{\mathsf{dLES}}$ in the cube.

Finally, we mention that not all squares (surfaces) of the 'cube' commute. Of course, they do with directions along those of the embeddings.

References

[1] M.A. Bednarczyk. *Categories of Asynchronous Transition Systems.* PhD Thesis, University of Sussex, 1988.

[2] P. Degano, R. Gorrieri, and S. Vigna. On Relating some Models for Concurrency. In Proceedings of *TAPSOFT '93*, LNCS n. 668, pp. 15–30, pp. 389–418, Springer-Verlag, 1993.

[3] M. Droste. Concurrent Automata and Domains. *Int. Journal of Foundations of Computer Science*, n. 3, pp. 389–418, 1992.

[4] M. Hennessy. *Algebraic Theory of Processes.* Cambridge, Massachusetts, 1988.

[5] C.A.R. Hoare. *Communicating Sequential Processes.* Englewood Cliffs, 1985.

[6] R.M. Keller. Formal Verification of Parallel Programs. *Communications of the ACM*, n. 19, vol. 7, pp. 371–384, 1976.

[7] S. MacLane. *Categories for the Working Mathematician.* GTM, Springer-Verlag, 1971.

[8] A. Mazurkiewicz. Basic Notions of Trace Theory. In *lecture notes for the REX summerschool in temporal logic*, LNCS n. 354, pp. 285–363, Springer-Verlag, 1988.

[9] R. Milner. *Communication and Concurrency.* Englewood Cliffs, 1989.

[10] M. Nielsen, G. Plotkin, and G. Winskel. *Petri nets, Event Structures and Domains, part 1.* Theoretical Computer Science, n. 13, pp. 85–108, 1981.

[11] M. Nielsen, V. Sassone, and G. Winskel. Relationships between Models of Concurrency. In *Proceedings of REX School '93. A Decade of Concurrency: Reflections and Perspectives*, LNCS n. 803, pp. 425–476, Springer-Verlag, 1994. Appears also as Technical Report DAIMI PB 456, University of Aarhus, 1994.

[12] C.A. Petri. *Kommunikation mit Automaten.* PhD thesis, Institut für Instrumentelle Mathematik, 1962.

[13] G. Plotkin. *A Structural Approach to Operational Semantics.* Technical report DAIMI FN-19, Computer Science Department, University of Aarhus, 1981.

[14] V. Sassone. *On the Semantics of Petri Nets: Processes, Unfoldings and Infinite Computations.* PhD Thesis, TD 6/94 Università di Pisa, March 1994.

[15] V. Sassone, M. Nielsen, and G. Winskel. A Classification of Models for Concurrency. In *Proceedings of Fourth International Conference on Concurrency Theory, CONCUR '93*, LNCS n. 715, pp. 82–96, Springer-Verlag, 1993.

[16] V. Sassone, M. Nielsen, and G. Winskel. Deterministic Behavioural Models for Concurrency. In *Proceedings of 18th International Symposium on the Mathematical Foundations of Computer Science, MFCS '93*, LNCS n. 711, pp. 682–692, Springer-Verlag, 1993.

[17] M.W. Shields. Concurrent Machines. *Computer Journal*, n. 28, pp. 449–465, 1985.

[18] E.W. Stark. Concurrent Transition Systems. *Theoretical Computer Science*, n. 64, pp. 221–269, 1989.

[19] G. Winskel. Event Structure Semantics of CCS and Related Languages. In *Proceedings of ICALP '82*, LNCS n. 140, pp. 561–567, Springer-Verlag, 1982. Expanded version available as technical report DAIMI PB-159, Computer Science Department, University of Aarhus.

[20] G. Winskel. Synchronisation Trees. *Theoretical Computer Science*, n. 34, pp. 33–82, 1985.

[21] G. Winskel. Event Structures. In *Advances in Petri nets*, LNCS n. 255, pp. 325–392, Springer-Verlag, 1987.

[22] G. Winskel, and M. Nielsen. Models for Concurrency. In *Handbook of Logic in Computer Science*, vol. 4, *Semantic Modelling*, S. Abramsky *et al.*, Eds, Oxford University Press, 1995.