# BDD’s

Developed from graphical representations of case arguments in propositional calculus.

Issue: Most representations of Boolean functions are “large”, i.e. for $t\left(\mathrm{B}^{n} \rightarrow \mathrm{B}^{1}\right)$, on n-bit Boolean function the representation of *t* is $O\left(2^{n}\right)$in general.

eg. Truth Table, K-map, SOP, CNF…

This is not surprising since each of the $2^{n}$ minterms of *t* can have an arbitrary value. However, in practice, we deal with Boolean functions with various types of symmetry.

Symmetry comes in many forms:

$$f\left(x_{1}, x_{2} \ldots x_{n}\right)=f\left(x_{2}, x_{1}, \ldots .\right) \qquad \text{(variable symmetry)}$$

$$f\left(x_{1}, x_{2} \ldots x_{n}\right)=\overline{f}\left(x_{2}, x_{1}, \ldots .\right) \qquad \text{(skew symmetry)}$$

$$f\left(x_{1}, x_{2} \ldots x_{n}\right)=g\left(x_{1}\right) \cdot f\left(x_{2}, \ldots . x_{n}\right) \qquad \text{(decomposition)}$$

(factoring by cubes)

Since we are often building circuits we expect that practical functions will have substantial composition symmetry:

$$f\left(x_{1}, x_{2}, \ldots x_{n}\right)=g\left(f_{1}\left(x_{1}, x_{2}, x_{3}\right), f_{2}\left(x_{4}, x_{5}, x_{6}, \ldots\right), f_{3}(\ldots)\right)$$

# BDD’s

Many kinds of symmetry are exposed by “case analysis”

i.e. def. $f(x_1,\ldots x_n)\big|_{x_1=1} \equiv f(1,x_2,\ldots x_n) = f\big|_{x_1=1} = f\big|_{x_1}$

as the $x_1 = 1$ <u>cofactor</u> of *f* $\qquad f(0,\ldots x_n) = f\big|_{\bar{x}_1}$

$f\big|_{x_1=1}$ can be thought of as *f* when $x_1 = 1$ or the “true case” for $x_1$

We have:

Thm: $\forall f(x_1,\ldots x_n),\ f(x_1,\ldots x_n) = x_1 \cdot f\big|_{x_1=1} + \overline{x}_1 \cdot f\big|_{x_1=0}$
(<u>Shannon Decomposition</u>)

pff: $f(x_1 \ldots x_n) = \Sigma$ (minterms)

3 cases: term = $\underbrace{x_1 \cdot (x_2,\ldots)}_{\| \atop t_1}$ ; $\underbrace{\bar{x}_1 \cdot (\ldots)}_{\| \atop t_2}$ ; $\underbrace{(\ldots)}_{\| \atop t_3}$

where $t_1, t_2, t_3$ do not involve $x_1$.

So we can write $f(x_1,\ldots x_n) = x_1 \cdot \Sigma(t_1) + \overline{x}_1 \cdot \Sigma(t_2) + \Sigma(t_3)$

# BDD's

now $$f(x_1,..,x_n)\big|_{x1} = \sum(t_1)+\sum(t_3), \quad f(x_1,..,x_n)\big|_{\bar{x}_1} \equiv \sum(t_2)+\sum(t_3)$$

$$f(x_1,..,x_n) = x_1 \cdot f\big|_{x_1} + \bar{x}_1 \cdot f\big|_{\bar{x}_1} \Rightarrow$$

$$x_1 \sum(t_1) + \bar{x}_1 \cdot \sum(t_2) + \underbrace{(x_1+\bar{x}_1)}_{\overset{//}{1}} \sum(t_3)$$

Case analysis is effective when many of the sub-functions produced by recursive Shannon decomposition are <u>equivalent</u>:

$$f(x_1,..,x_n) = x_1 x_2 \bar{x}_3 f(\ )\big|_{x_1 x_2 \bar{x}_3} + x_1 \bar{x}_2 x_3 f(\ )\big|_{x_1 \bar{x}_2 x_3} + \ldots$$

and we have for some cases: $f(\ )\big|_{x_1 x_2 \bar{x}_3} = f(\ )\big|_{x_1 \bar{x}_2 x_3} \ldots$

$\Rightarrow$# of cases grows far slower than $2^k$ at stage $k$.

eq. consider $s(x_1,..,x_n)$ totally symmetric function $\Rightarrow$

$s(x_1,..,x_n) = s(x_{\pi_1},\ldots x_{\pi_n})$ for <u>any</u> permutation $(\pi_1 \ldots \pi_n)$

eq. $$s(x_1,..,x_n) = x_1 x_2 s\big|_{x_1 x_2} + x_1 \bar{x}_2 s\big|_{x_1 \bar{x}_2} + \bar{x}_1 x_2 s\big|_{\bar{x}_1 x_2} + \bar{x}_1 \bar{x}_2 s\big|_{\bar{x}_1 \bar{x}_2}$$

# BDD's

but we know that $s(x_1, x_2 \ldots) = s(x_2, x_1 \ldots)$ so…

$\Rightarrow s\big|_{x_1\bar{x}_2} = s\big|_{\bar{x}_1 x_2}$

$= x_1 x_2 s\big|_{x_1 x_2} + (x_1\bar{x}_2 + \bar{x}_1 x_2) s\big|_{x_1\bar{x}_2} + \bar{x}_1\bar{x}_2 s\big|_{\bar{x}_1\bar{x}_2}$

In general: $s(x_1, \ldots x_n) = \sum (x_1, .., x_n) \cdot s\big|_{(x_1 \ldots x_n)}$

but from symmetry, $s\big|_{\overline{x_1\bar{x}_2 \ldots x_n}} = s\big|_{\overline{\bar{x}_1 x_2 \ldots x_n}}$ if same #of true & false variables

$\Rightarrow s(x_1 \ldots x_n) = x_1 \ldots x_n s\big|_{x_1 \ldots x_n} + (x_1 x_2 \ldots \bar{x}_n + (\iota \ldots \iota)\bar{x}_1 x_2 \ldots x_n)$

$+(x_1 \ldots \bar{x}_{n-1}\bar{x}_n + \ldots + \bar{x}_1\bar{x}_2 x \ldots x_n) \cdot s\big|_{x_1 \ldots \bar{x}_{n-1}\bar{x}_n}$

+(3 false terms)$\cdot s\big|_{x_1 \ldots \bar{x}_{n-2}\bar{x}_{n-1}\bar{x}_n}$

+(4 “)·

+…

$+\bar{x}_1 \cdot \bar{x}_2\bar{x}_3 \ldots \bar{x}_n \cdot s\big|_{\bar{x}_1\bar{x}_2 \ldots \bar{x}_n}$

However each $s\big|_{x_1 \ldots x_n}$ term is <u>constant</u> since it does not depend on $x_1 \ldots x_n$

Let $P_{n,m} = \sum_{\text{all } i} x_{i_1} x_{i_2} (\overline{\ldots x_{i_n}})$
m terms false

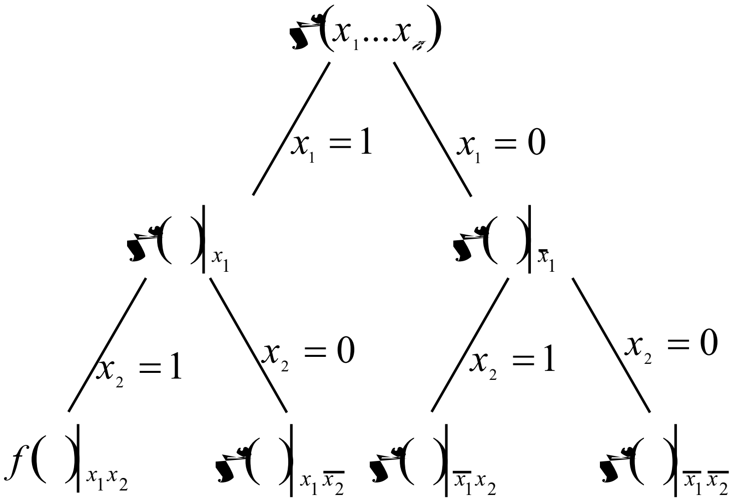
then any $s(x_1 \ldots x_n)$ totally symmetric $= \sum_{i=1}^{n} P_{n,i} \cdot S_i$ for some $n$, $S_i$ constants.

i.e. for 10-variables, ∃ only 1024 different totally symmetric functions.

# BDD’s

Can we find a representation for $s(x_1 \ldots x_n)$ which <u>naturally</u> “finds” such symmetry?

Lets graph $f(x_1 \ldots x_n)$’s decomposition:

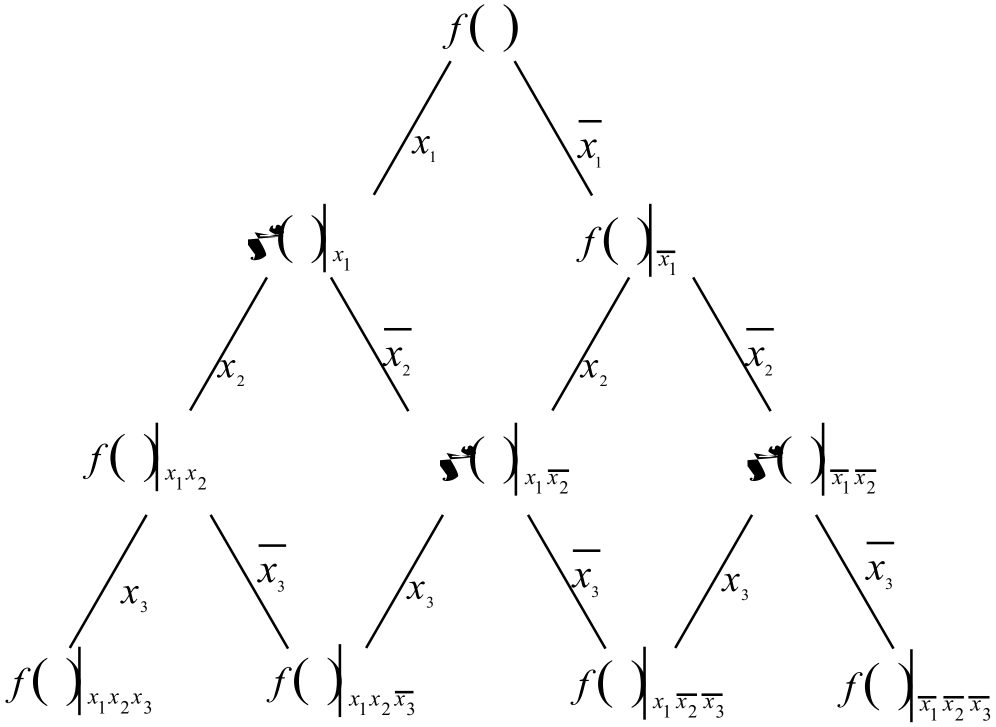


for our symmetric case, $f(\ )\big|_{x_1 \overline{x_2}} = f(\ )\big|_{\bar{x}_1 x_2}$, <u>in general</u> we might have only a subset of $f(\ )\big|_{x_1 x_2 \ldots x_x}$ be unique…

# BDD’s

So we only add a single node to represent each different $f\big|_{x_1 \dots x_x}$.

eg:

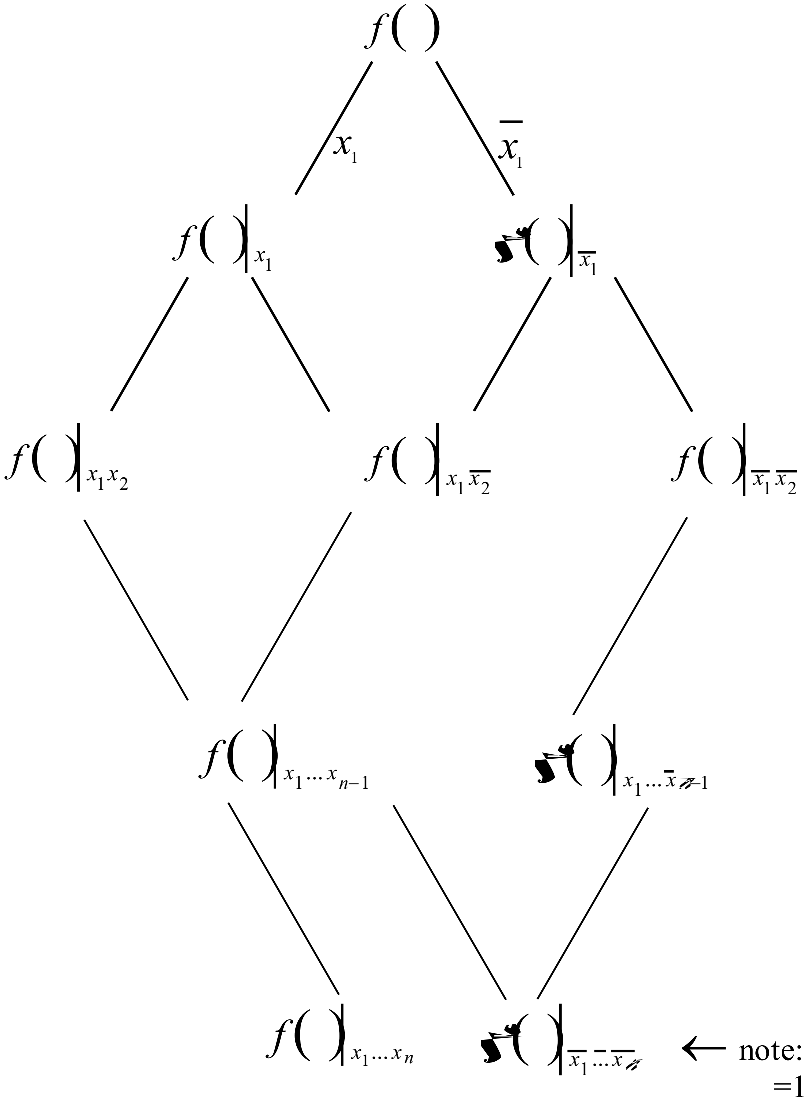


$f(\;)$
$x_1$
$\overline{x_1}$
$f(\;)\big|_{x_1}$
$f(\;)\big|_{\overline{x_1}}$
$x_2$
$\overline{x_2}$
$x_2$
$\overline{x_2}$
$f(\;)\big|_{x_1 x_2}$
$f(\;)\big|_{x_1 \overline{x_2}}$
$f(\;)\big|_{\overline{x_1}\,\overline{x_2}}$
$x_3$
$\overline{x_3}$
$x_3$
$\overline{x_3}$
$x_3$
$\overline{x_3}$
$f(\;)\big|_{x_1 x_2 x_3}$
$f(\;)\big|_{x_1 x_2 \overline{x_3}}$
$f(\;)\big|_{x_1 \overline{x_2}\,\overline{x_3}}$
$f(\;)\big|_{\overline{x_1}\,\overline{x_2}\,\overline{x_3}}$

note: It is <u>not</u> easy in general to determine if $f(\;)\big|_{ab\bar{c}\bar{d}} = f(\;)\big|_{\bar{a}cd\bar{b}}$. However it is trivial to tell if $f(x_1 \dots x_n)\big|_{\bar{x}_1 \dots \bar{x}_n} \overset{?}{=} f(x_1 \dots x_n)\big|_{x_1 \dots \bar{x}_n}$ since both are constants.

# BDD's

So if we recursively decompose $f(\ )$ on $x_1 \ldots x_n$, and at each step collapse all equivalent functions we arrive at a graph: OBDD $\left(f(\ ), x_1 \ldots x_n\right)$:



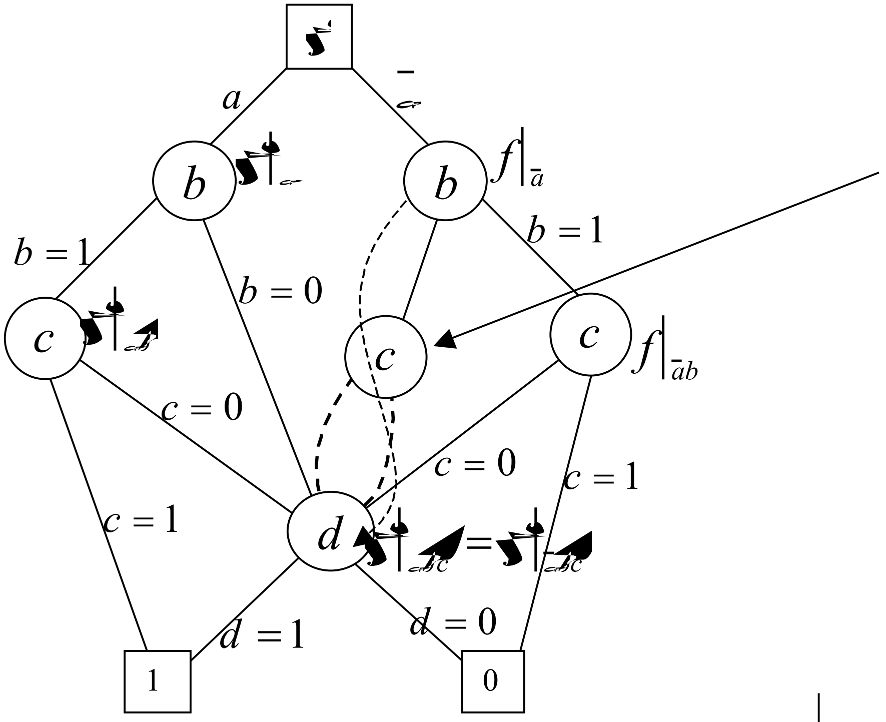
$f(\ )$
$x_1$
$\overline{x_1}$
$f(\ )\big|_{x_1}$
$f(\ )\big|_{\overline{x_1}}$
$f(\ )\big|_{x_1 x_2}$
$f(\ )\big|_{x_1 \overline{x_2}}$
$f(\ )\big|_{\overline{x_1}\,\overline{x_2}}$
$f(\ )\big|_{x_1 \ldots x_{n-1}}$
$f(\ )\big|_{x_1 \ldots \overline{x}_{n-1}}$
$f(\ )\big|_{x_1 \ldots x_n}$
$f(\ )\big|_{\overline{x_1} \ldots \overline{x_n}}$

← note: only two cases here!
=1,0

So: How to build this?

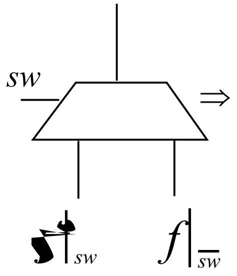
# BDD’s

Ex: $f = abc + \bar{b}d + \bar{c}d$



note: if both edges from a node go to the <u>same</u> node: remove $\Longrightarrow$ redundant.

- Each node could both be though of as a <u>mux</u>



$\Rightarrow$ BDD size <u>bounds circuit size</u>

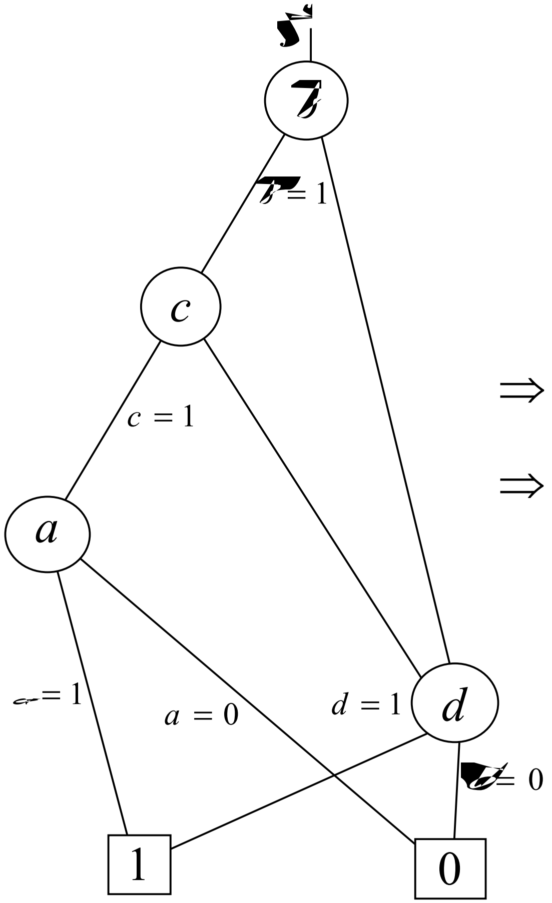
# BDD's

- ⇒# of paths through BDD from f to 1 terminal bounds # of terms of SOP for $f$.

$$f = abc + a\bar{b}d + ab\bar{c}d + \bar{a}\bar{b}d + \bar{a}b\bar{c}d$$

corr⇒ # of paths from $f$ to $0$ terminal bounds # of terms $|SOP|$

Note: same $f$ :



⇒ Each path from $f$ to 1 is a cube of $f$

⇒ paths ≥ $|ISOP|$

⇒ only split on a variable one time in path to terminal

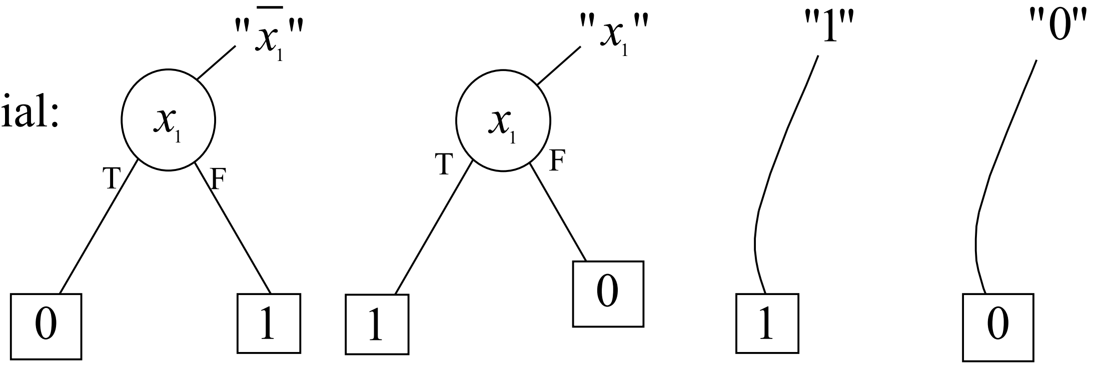
⇒ size/shape of BDD is ORDER Dependent!

# BDD's

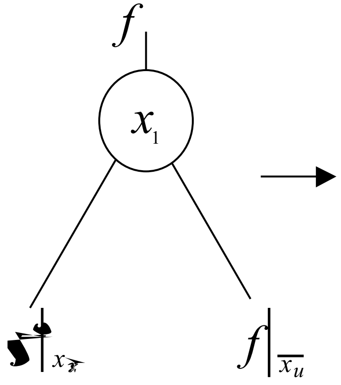
*thm*: for a fixed variable order, *ROBDD* is canonical; i.e. logic function *f* is represented by a *ROBDD* in only one way (isomorphic to graph of *f* built any other way).

pff: (Induction on *n*):

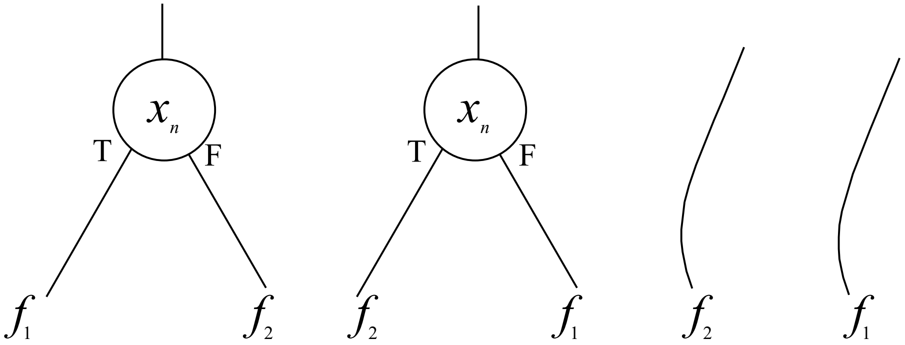
case $n=1$ ⇒ trivial:



Assume true for $n-1$ ⇒



only four cases:



# BDD’s

Back to Symmetric Case:

For symmetric $f(x_1 \dots x_n)$, maximal # of terms is only $\underline{\underline{n}}$
for any level $\Rightarrow$ BDD size $O(n^2)$
So BDD naturally exploits same kinds of symmetries.