BDDs:
Implementation Issues
&
Variable Ordering

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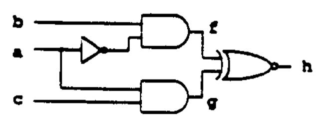
OBDD Package Interface

```
typedef struct bdd_manager_struct *bdd_manager;
typedef struct bdd_formula_struct *bdd_formula;

bdd_manager bdd_start();
void bdd_end(bdd_manager bdd);

bdd_formula bdd_create_variable(bdd_manager bdd);
bdd_formula bdd_zero(bdd);
bdd_formula bdd_one(bdd);
bdd_formula bdd_assign(bdd_formula f);
bdd_formula bdd_not(bdd_formula f);
bdd_formula bdd_and(bdd_formula f, bdd_formula g);
bdd_formula bdd_or(bdd_formula f, bdd_formula g);
bdd_formula bdd_xor(bdd_formula f, bdd_formula g);
bdd_formula bdd_ite(bdd_formula f, bdd_formula g, bdd_formula h);
bdd_formula bdd_exists(bdd_formula f, bdd_formula g);
bdd_formula bdd_forall(bdd_formula f, bdd_formula g);
bdd_formula bdd_cofactor(bdd_formula f, bdd_formula g);
bdd_formula bdd_compose(bdd_formula f, bdd_formula g, bdd_formula h);
void bdd_free(bdd_formula f);

int bdd_equ(bdd_formula f, bdd_formula g);
int bdd_leq(bdd_formula f, bdd_formula g);
int bdd_disjoint(bdd_formula f, bdd_formula g);
int bdd_ite_tautology(bdd_formula f, bdd_formula g, bdd_formula h);
int bdd_sat(bdd_formula f, bdd_formula *lits);
```


OBDD Package Example

```
bdd_manager bdd;
bdd_formula a, b, c, f, g, h, anot, one;

bdd = bdd_start();
one = bdd_one(bdd);

a = bdd_create_variable(bdd);
b = bdd_create_variable(bdd);
anot = bdd_not(a);
f = bdd_and(anot, b);
bdd_free(anot);
c = bdd_create_variable(bdd);
g = bdd_and(a, c);
h = bdd_xnor(f, g);
if (bdd_equ(h, one)) {
     << do something >>
}
bdd_free(f);
bdd_free(g);
bdd_free(h);

bdd_end(bdd);
```



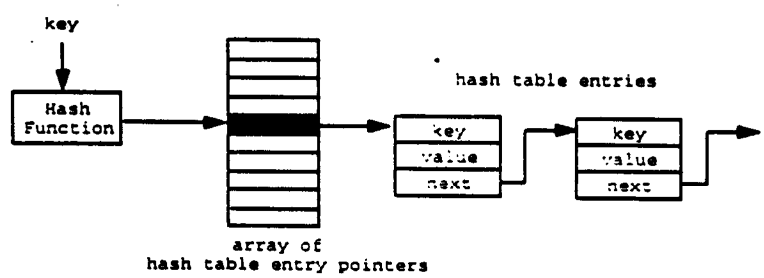
Review: Chained Hash Table

Insert the pair (key, value) into the hash table

```
void hash_insert(hash_table *hash_table, void *key, void *value);
```

Return the value associated with the given key (if it exists)

```
int hash_lookup(hash_table *hash_table, void *key, void **value);
```



- density = # entries / # bins
- resize array to maintain constant density (e.g., 4 entries per bin)
- constant-time lookup operation (assuming good hash function)

Review: Memory Function

- Store table of values $(x, F(x))$ for a pure function F.
- Before computing $F(x)$, check table for stored value
  - avoid re-computing $F(x)$ if value is already known
  - when $F(x)$ is computed, save $(x, F(x))$ in the table

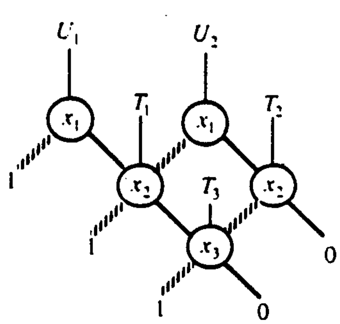
no memory function, exponential complexity

```
int fib(int n) {
    int t;

    if (n <= 2) {
        t = 1;
    } else {
        t = fib(n-1) + fib(n-2);
    }
    return t;
}
```

with memory function, linear complexity

```
int fib1(int n) {
    static int memory[100];
    int t;

    if (n <= 2) {
        t = 1;
    } else if (memory[n] != 0) {
        t = memory[n];
    } else {
        t = fib1(n-1) + fib1(n-2);
        memory[n] = t;
    }
    return t;
}
```


Multi-Rooted (Shared) OBDD

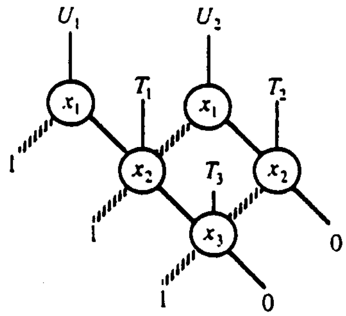


$$\left.\begin{aligned} U_1 &= \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = (x_1, T_1, 1) \\ U_2 &= \bar{x}_1\bar{x}_2 + \bar{x}_1\bar{x}_3 = (x_1, T_2, T_1) \end{aligned}\right\} \text{External functions}$$

$$\left.\begin{aligned} T_1 &= \bar{x}_2 + \bar{x}_3 = (x_2, T_3, 1) \\ T_2 &= \bar{x}_2\bar{x}_3 = (x_2, 0, T_3) \\ T_3 &= \bar{x}_3 = (x_3, 0, 1) \\ 0 &= (x_\infty, \circ, \circ) \\ 1 &= (x_\infty, \bullet, \bullet) \end{aligned}\right\} \text{Internal Functions}$$

- A DAG node F is represented by a tuple (x_i, G, H)
  - x_i is called the *top variable* of F
  - node (x_i, G, H) represents the function $\text{ite}(x_i, G, H) = x_i G + \bar{x}_i H$
- DAG contains both *external functions (user functions)* and *internal functions*

Unique Table



Hash Table Mapping

$(x_1, T_1, 1) \to U_1$
$(x_1, T_2, T_1) \to U_2$
$(x_2, T_3, 1) \to T_1$
$(x_2, 0, T_3) \to T_2$
$(x_3, 0, 1) \to T_3$
$(x_\infty, \bullet, \bullet) \to 1$
$(x_\infty, \circ, \circ) \to 0$

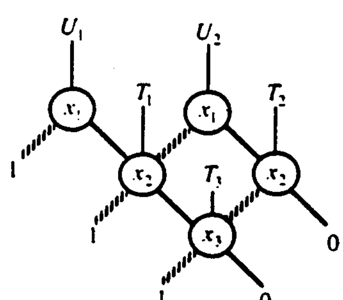
- **Unique table**: hash table mapping tuples(x_i, G, H) into a node in the DAG
  - before adding a node to the DAG, check to see if it already exists
  - avoids ever creating two nodes with the same function
  - strong canonical form: pointer equality determines function equality
  - resize unique table array to maintain constant density of 4 entries/bin

Shannon Cofactors of an OBDD Function

Computing the *Shannon Cofactor* (Restriction) on an OBDD function is trivial when the variable is *at or above* the top variable of the node.

Let $F = (x_i, G, H)$ and let x_j be a variable at level i or above (i.e., $j \le i$). Then,

$$F_{x_j} = \begin{cases} F & \text{if } j < i \\ G & \text{if } j = i \end{cases} \quad \text{and} \quad F_{\bar{x}_j} = \begin{cases} F & \text{if } j < i \\ H & \text{if } j = i \end{cases}$$



x_2 is top variable of T_1

$(T_1)_{x_2} = T_1|_{x_2=1} = T_3$

$(T_1)_{\bar{x}_2} = T_1|_{x_2=0} = 1$

x_1 is above top variable of T_1

$(T_1)_{x_1} = T_1|_{x_1=1} = T_1$

$(T_1)_{\bar{x}_1} = T_1|_{x_1=0} = T_1$

ITE Recursive Formulation

Let $Z = \text{ite}(F,G,H) = FG + \bar{F}H$. Let x be the top variable of F,G,H.

$$
\begin{aligned}
Z &= xZ_x + \bar{x}Z_{\bar{x}} \\
&= x(FG + \bar{F}H)_x + \bar{x}(FG + \bar{F}H)_{\bar{x}} \\
&= x(F_xG_x + \bar{F}_xH_x) + \bar{x}(F_{\bar{x}}G_{\bar{x}} + \bar{F}_{\bar{x}}H_{\bar{x}}) \\
&= \text{ite}(x, \text{ite}(F_x,G_x,H_x), \text{ite}(F_{\bar{x}},G_{\bar{x}},H_{\bar{x}})) \\
&= (x, \text{ite}(F_x,G_x,H_x), \text{ite}(F_{\bar{x}},G_{\bar{x}},H_{\bar{x}}))
\end{aligned}
$$

$\therefore \text{ite}(F,G,H) = (x, \text{ite}(F_x,G_x,H_x), \text{ite}(F_{\bar{x}},G_{\bar{x}},H_{\bar{x}}))$.

Because x is the top variable of F,G,H, the cofactors $F_x, F_{\bar{x}}$, etc. are trivial

Terminal cases:

$\text{ite}(1,G,H) = G$
$\text{ite}(0,G,H) = H$
$\text{ite}(F,1,0) = F$

Computed Table

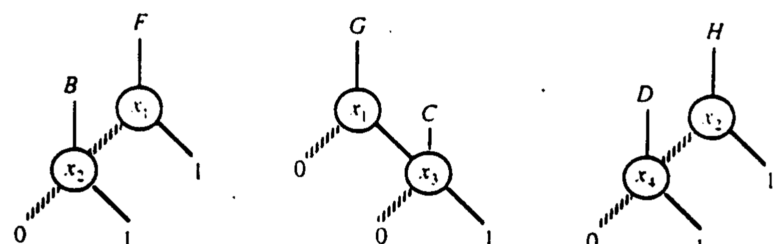
- **Computed table**: hash table to implement a memory function for ITE
  - Maps ITE arguments (F,G,H) into the result $\text{ite}(F,G,H)$

- Computed table is persistent
  - Computed table results remain valid across top-level calls to ITE
  - allows results computed from previous ITEs to improve performance of subsequent ITEs
  - no need to initialize and free the computed table every ITE
    initialize computed table once when the OBDD is created
    saves linear time cost of allocating and freeing the table every ITE

ITE Algorithm

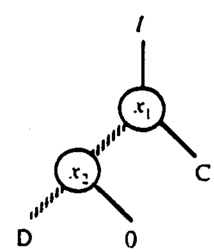
```
ite(F,G,H) {
   if (terminal case) {
      R = trivial answer;
   } else if (hash_lookup(computed_table, (F,G,H), &result)) {
      R = result;
   } else {
      x = top variable from F, G, H;
      (F1,F0) = trivial_cofactor(F, x);
      (G1,G0) = trivial_cofactor(G, x);
      (H1,H0) = trivial_cofactor(H, x);
      R1 = ite(F1,G1,H1);
      R0 = ite(F0,G0,H0);
      if (R1 == R0) {
         R = R1;
      } else if (hash_lookup(unique_table, (x,R1,R0), &result)) {
         R = result;
      } else {
         R = new_node(x,R1,R0);
         hash_insert(unique_table, (x,R1,R0), R);
      }
      hash_insert(computed_table, (F,G,H), R);
   }
   return R;
}
```


ITE Algorithm Trace



$F = x_1 + x_2$ $G = x_1 x_3$ $H = x_2 + x_4$

$$
\begin{aligned}
I &= \text{ite}(F, G, H) \\
&= \left(x_1, \text{ite}(F_{x_1}, G_{x_1}, H_{x_1}), \text{ite}(F_{\bar{x}_1}, G_{\bar{x}_1}, H_{\bar{x}_1})\right) \\
&= \left(x_1, \text{ite}(1, C, H), \text{ite}(B, 0, H)\right) \\
&= \left(x_1, C, (x_2, \text{ite}(B_{x_2}, 0_{x_2}, H_{x_2}), \text{ite}(B_{\bar{x}_2}, 0_{\bar{x}_2}, H_{\bar{x}_2})\right) \\
&= \left(x_1, C, (x_2, \text{ite}(1, 0, 1), \text{ite}(0, 0, D)\right) \\
&= \left(x_1, C, (x_2, 0, D)\right)
\end{aligned}
$$



ITE Algorithm Improvements

- Improve computed table performance - equivalent forms

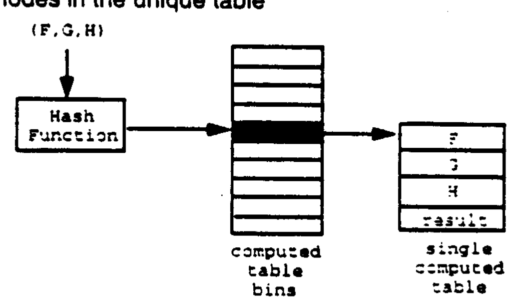
$$\text{ite}(F,G,0) = \text{ite}(G,F,0) = \text{ite}(F,G,F) = \text{ite}(G,F,G) = FG$$
$$\text{ite}(F,1,H) = \text{ite}(H,1,F) = \text{ite}(F,F,H) = \text{ite}(H,H,F) = F+H$$

  - store only 1 of 4 equivalent forms in the computed table
    map to a canonical form (e.g., ite(F,G,0) with addr(F) < addr(G))
  - easy to detect because of strong canonical form

```
if (F == H || H == 0) {
    /* function is F G */
    H = 0;
    if (address(F) > address(G)) {
        swap(&F,&G);
    }
}
if (F == G || G == 1) {
    /* function is F + H */
    G = 1;
    if (address(F) > address(H)) {
        swap(&F,&H);
    }
}
```


Computed Table Cache

- Replace computed table hash table with a hash-based cache
  - store only one entry per bin (no collision chain)
    overwrite existing entry at insert
    check against only one entry at lookup
  - introduces possibility of *cache miss* which forces redundant computation (affects performance, but not correctness)
    manage impact by sizing the cache proportional to the number of nodes in the unique table



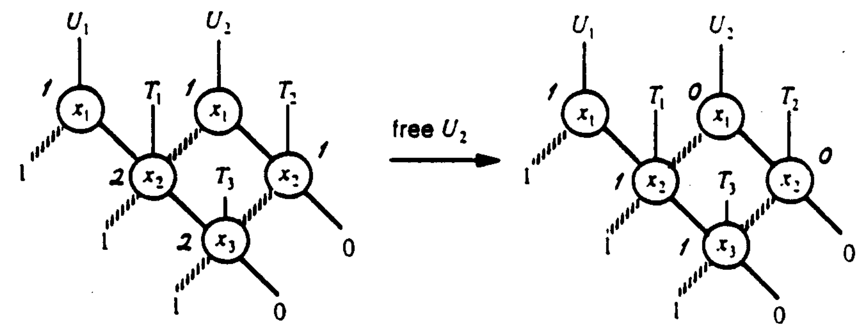
Reusing Memory

- New nodes are added to the DAG during ITE
  - minimum number of nodes to represent the result are created!
- The user discards old computation results using `bdd_free()`
  - problems with deleting the nodes immediately
    1. need to know if the nodes are shared by other roots
    2. computed table entries are never deleted
    3. computed table entries may point at the node
       back-pointers would take too much memory
       sweeping entire computed table would be too slow

Garbage Collection

- Solution - Garbage Collection
  - maintain reference count for each node
    includes user references and internal references
    does not count references from the computed table
    reference count is incremented when nodes are reused in the DAG
    reference count is decremented when a root is freed by the user
  - nodes with reference count of 0 are called *dead*
    they remain in the DAG until the next garbage collection
  - periodic garbage collection
    delete all computed table entries which point to a dead node
    remove all dead nodes from the unique table

Reference Counting Example

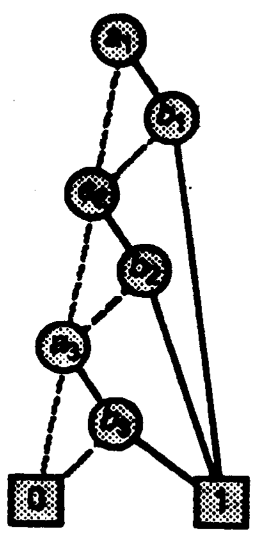


- Freeing formula U_2 reduces reference count on nodes below U_2
  - reduce count of U_2 to 0; it becomes dead so free its children
  - reduce count of T_1 to 1
  - reduce count of T_2 to 0; it becomes dead so free its children
  - reduce count of T_3 to 1
- Nodes U_2 and T_2 have ref count 0
  - they will be made available for re-use at the next garbage collection

Effect of Variable Ordering

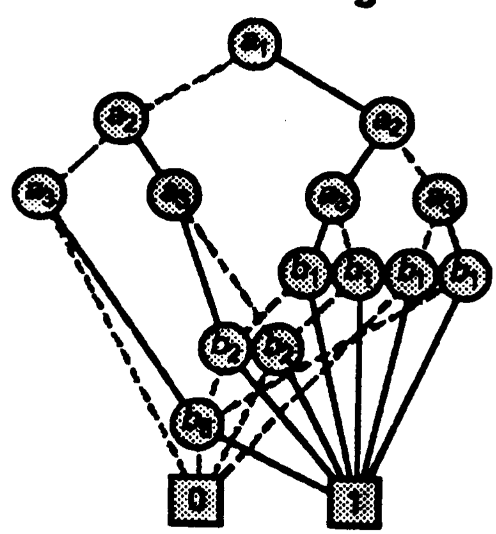
$$a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$$

Good Ordering



Linear Growth

Bad Ordering



Exponential Growth

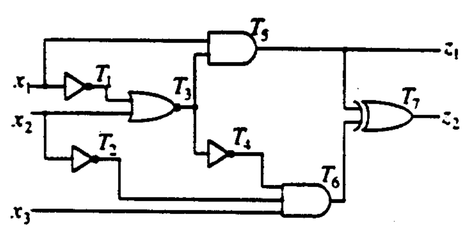
OBDD Variable Ordering

- Goal: Form OBDD functions for all nets of a combinational circuit
  - represent function of every net in terms of primary inputs called the *global functions*
  - first step of verification and optimization algorithms
- OBDDs for all nets or just primary outputs?
  - comb. and seq. verification require only primary output OBDDs
  - optimization algorithms require OBDDs for all nets
- Consistent variable order for all nets?
  - comb. verification can handle different order for each primary output
  - seq. verification and optz algorithms need same order for all nets
- Why worry about variable ordering?
  - using a random variable order almost always fails

    e.g., OBDDs cannot be formed for 23 of the 35 largest circuits from IWLS'91 benchmark set when using a randomly generated order and 100,000 node limit

OBDDs for Combinational Circuits

- Depth-first walk on combinational circuit from each primary output
  - form logic function for net in terms of primary inputs only



$$
\begin{aligned}
T_1 &= \bar{x}_1 \\
T_3 &= T_1 x_2 = \bar{x}_1 x_2 \\
T_5 &= x_1 T_3 = 0 \\
z_1 &= T_5 = 0 \\
T_4 &= \bar{T}_3 = x_1 + \bar{x}_2 \\
T_2 &= \bar{x}_2 \\
T_6 &= T_4 T_2 x_3 = x_1 \bar{x}_2 x_3 + \bar{x}_2 x_3 \\
T_7 &= T_5 \oplus T_6 = x_1 \bar{x}_2 x_3 + \bar{x}_2 x_3 \\
z_2 &= T_7 = x_1 \bar{x}_2 x_3 + \bar{x}_2 x_3
\end{aligned}
$$

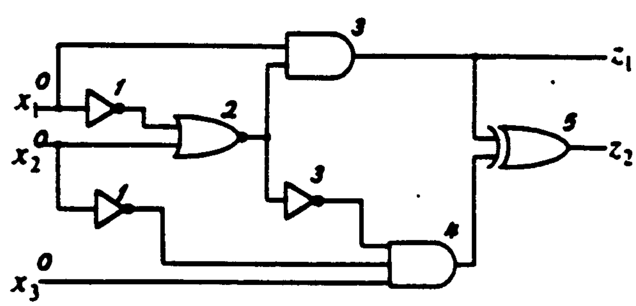
Heuristic Variable Order

- Use circuit topology to find a good variable order
  - [Fujita-ICCAD88], [Malik-ICCAD88], [Minato-DAC90]
  - variations on the following idea
- Define depth of each node n:

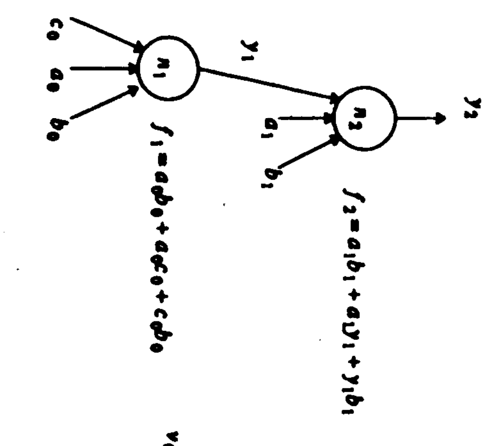
$$d(n) = \begin{cases} \max_{f \in FI(n)} d(f) + 1 & \text{if } n \text{ not a primary input} \\ 0 & \text{if } n \text{ is a primary input} \end{cases}$$

- Starting from deepest output, traverse network in depth-first fashion
  - order fanin at each node by decreasing depth
    explore deep fanins first
  - break ties arbitrarily (or with more heuristics)
- Order of traversal of primary inputs defines OBDD variable order
  - first variable visited is at the top of the OBDD

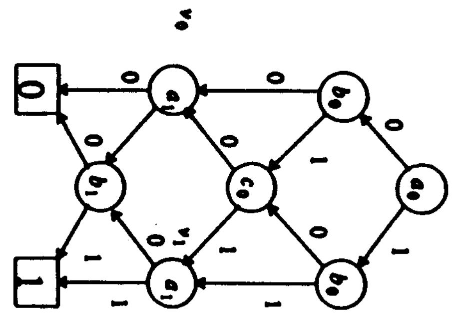
Heuristic Variable Order Example



- Deepest output is z_2
- Depth-first traversal, ordered by depth, visits inputs in order:
  x_1, x_2, x_3



Multi-Level Network



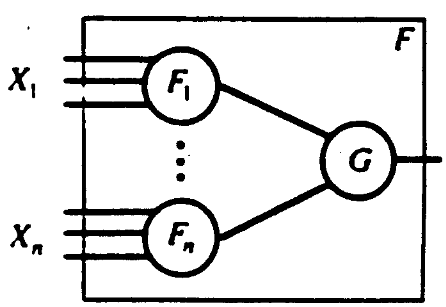
BDD

```
/* level heuristic */
for each node n in η_n
    compute level(n);
order_list = nodes sorted in decreasing levels;
```

Rationale for Depth-First Heuristic

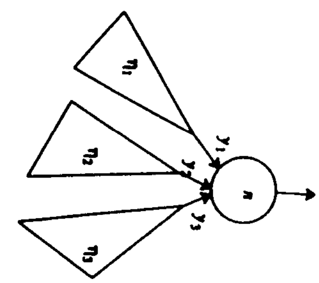
- Primary inputs which feed deep cones of logic get ordered near the top
  - heuristic: they are the more important decision makers

- **Theorem:** If $F = G(F_1, F_2, \ldots, F_n)$ where each pair of functions F_i and F_j $(i \neq j)$ share no variables in common, then there exists an optimum OBDD variable order for F which consists of a noninterleaved concatenation of the optimum variable orders of the F_i (for some ordering of the functions $F_1, F_2, \cdots F_n$).



X_i is optimum variable order for F_i

Optimum variable order for F is
$(X_{\sigma(1)}, X_{\sigma(2)}, \cdots, X_{\sigma(n)})$
for some permutation σ.



Lemma 1 *If a function f can be written in the form:*

$$f = g(f_1, g_1(f_2, g_2(\cdots g_{n-1}(f_n, f_{n+1})\cdots)))$$

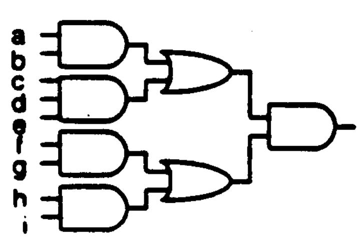
where the g_is are any two argument Boolean functions, and each f_i has support that is disjoint from that of the others, then the optimum ordering for f is the concatenation of the optimum orderings for the f_is in the order 1, 2, ... n+1.

```
/* fanin heuristic */
order_list = null;
faninOrder(n, order_list);

faninOrder(node, order_list)
{   if(node ∉ order_list)
        foreach fanin
            compute TFI DAG depth;
        sorted_fanin_list = fanins sorted in
        decreasing TFI DAG depths;
        foreach fanin in sorted_fanin_list
            faninOrder(fanin, order_list);
        append(order_list, node);
}
```

Special Case: Fanout Free Circuits

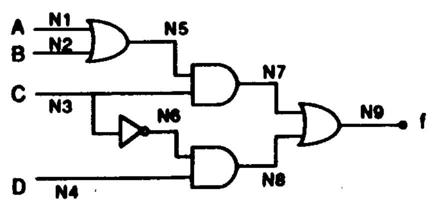
- **Theorem:** If G is AND or OR, then there exists an optimum variable order for F which consists of an arbitrary (order-independent) noninterleaved concatenation of the optimum variables orders of the F_i.

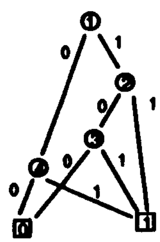
- **Corollary:** The depth-first ordering algorithm returns an optimum order for a combinational circuit with no reconvergent fanout composed from simple gates (AND, OR, NOT).



Optimum orders:
(a b)(c d e)(f g)(h i)
(c d e)(a b)(h i)(f g)
(i h)(g f)(e d c)(b a)


```
{ Initially all nets must be marked off }
procedure makeOrder(N);
begin
   foreach I < Set of all input nets of the gate to which N is connected do
   begin
      if I is marked then continue;
      if I is directly connected to a primal input then
         if I is connected more than one gate then
         begin
            FANOUT2UP := I;
            if I is not in ORDER then ORDER := append(ORDER,I);
         end else FANOUT1LIST := append(FANOUT1LIST, I);
      else makeOrder(I);
   end;
   if FANOUT2UP <> undef then
   begin
      Insert FANOUT1LIST into ORDER after FANOUT2UP;
      FANOUT1LIST := NIL;
   end;
   Mark N;
   return;
end;
```





traverse of circuit	FANOUT2UP	FANOUT1LIST	ORDER
N9	undef	NIL	NIL
N7	undef	NIL	NIL
N5	undef	NIL	NIL
N1	undef	A	NIL
N2	undef	A,B	NIL
N3	C	NIL	C
N8	C	NIL	C,A,B
N6	C	NIL	C,A,B
N4	C	D	C,A,B
finish	C	NIL	C,A,B,D

Name	Inputs	Outputs	Levels	Gates	for single output	for single output
SN181	14	8	9	34	55	14
c432	36	7	7	203	145	36
c499	41	32	11	275	102	41
c880	60	26	24	469	130	45
c1355	41	32	24	619	322	41
c1908	33	25	40	938	557	33
c2670	233	140	32	1566	828	122
c3540	50	22	47	1741	1433	50
c5315	178	123	49	2608	937	67
c6288	32	32	124	2480	2327	32
c7552	207	108	43	3827	1096	194

Name	Original order		Manual order		Random order		Our algorithm order	
	Time	Maximum nodes	Time	Maximum nodes	Time	Maximum nodes	Time	Maximum nodes
SN181	8	339	7	197	8	578	6	213
c432	109	1146	53	442	unable	>100000	1423	6196
c499	928	9020	673	4661	unable	>100000	673	4661
c880	unable	>100000	85	3421	unable	>100000	55	3359
c1355	675	. 9020	—	—	unable	>100000	1009	4661
c1908	965	2912	1800	4505	>50000	—	778	3076
c2670	unable	>100000	—	—	unable	>100000	338	14763
c3540	unable	>100000	—	—	unable	>100000	12620	53460
c5315	2140	11807	—	—	unable	>100000	498	3441
c6288	unable	>100000	—	—	unable	>100000	unable	>100000
c7552	unable	>100000	—	—	unable	>100000	383	2096

Ckt.	level separate		fanin separate		level together		fanin together	
	Time	Max. BDD	Time	Max. BDD	Time	Max. BDD	Time	Max. BDD
C432	1232	7993	1225	6881	617	7993	530	6881
C499	1476	6779	1102	6597	599	6779	394	6741
C880	374	5840	205	3102	270	5069	-	-
C1355	8986	6779	4995	6597	2064	6779	1198	6741
C1908	10059	5902	6016	3092	1388	8022	735	4037
C2670	-	-	4702	796	-	-	-	-
C3540	-	-	23580	68341	-	-	-	-
C5315	-	-	7942	3248	-	-	-	-
C6288	-	-	-	-	-	-	-	-
C7552	-	-	6888	1299	-	-	-	-
rot	414	4013	400	1225	240	2390	238	4272
des	1773	340	1509	86	4594	412	4591	591

Heuristic Variable Ordering Limitations

- Random orders almost always fail
  - fails for 23 of 35 largest examples in IWLS'91 benchmark set
- Depth-first heuristic order also fails for many examples
  - fails for 11 of 35 largest examples in IWLS'91 benchmark set
- Is this inherent OBDD exponential complexity or just bad orders?
- Many functions exhibit behavior that some orders produce large OBDDs while other orders produce small OBDDs
  - e.g., n-bit adder

    $(a_{n-1}, b_{n-1}, a_{n-2}, b_{n-2}, \cdots, a_1, b_1, a_0, b_0)$ linear

    $(a_{n-1}, a_{n-2}, \cdots a_1, a_0, b_{n-1}, b_{n-2}, \cdots, b_1, b_0)$ exponential
  - e.g., Achilles Heel function: $f = x_0 x_1 + x_2 x_3 + \cdots + x_{n-2} x_{n-1}$

    $(x_0, x_1, x_2, x_3, \cdots, x_{n-2} x_{n-1})$ linear

    $(x_0, x_2, x_4, \cdots, x_{n-2}, x_1, x_3, \cdots, x_{n-1})$ exponential

Dynamic Variable Ordering

- Motivation:
  - many OBDD operations run out of memory using heuristic ordering
  - programmer must devise application-specific ordering algorithms for each OBDD application

    can be complex for some applications

    expend effort on better heuristics or finding a non-OBDD solution?
- Solution: **Dynamic Variable Ordering**
  - allow the OBDD package to modify the order *on the fly*

    OBDD package hides all variable ordering details from user

    OBDD order is no longer static

    Allow OBDD order to change in-between operations
  - maintain consistent order for the OBDD before & after each ITE

    modify the order as a side-affect of OBDD processing

    use current OBDD functions to determine new variable order

Dynamic Variable Ordering Paradigm

- General solution paradigm with many choices
  - when to modify the order?
    e.g., every time the OBDD DAG doubles in size
    e.g., when memory limit is exceeded
    e.g., every 10 ite operations
    e.g., every 100,000 ite steps
  - how to choose a new order
    e.g., variety of OBDD minimization algorithms

- Logically perform variable ordering in-between operations, but:
  - F and G are small, but $F+G$ is too large to be represented
    need to make all 3 functions ($F, G, F+G$) small simultaneously
  - solution: reorder variables deep in ITE recursion
    include partial result for $F+G$

OBDD Minimization

- **Problem definition:** Given a multi-rooted OBDD DAG, reorder the variables to minimize the number of nodes in the DAG needed to represent all user functions (simultaneously)

- Complexity:
  $n!$ permutations (orders) for n variables
  brute-force search: $O(n!2^n)$
  dynamic programming search: $O(n^2 3^n)$ [Friedman & Supowit]

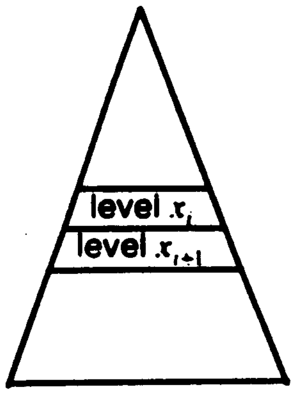
- Optimum ordering problem is NP-hard

- Don't need optimum order!
  Just want to avoid exponentially-sized worst case when possible

Adjacent Variable Swap

- Swapping the order of two adjacent variables
  - affects only the nodes at the two levels!

- For a single OBDD function F, the nodes at level i represent the unique functions from the set $\left\{F_{\bar{x}_1\bar{x}_2\cdots\bar{x}_{i-1}}, F_{x_1\bar{x}_2\cdots\bar{x}_{i-1}}, \cdots, F_{x_1x_2\cdots x_{i-1}}\right\}$ which depend on x_i

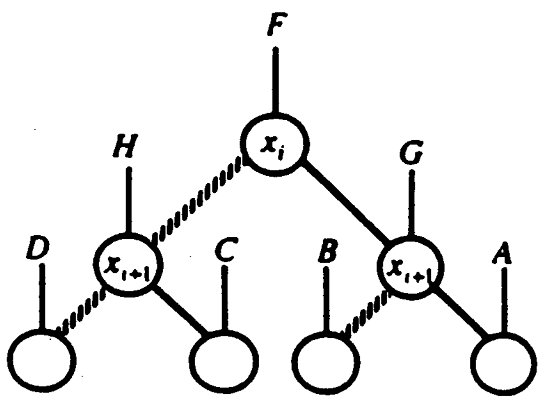


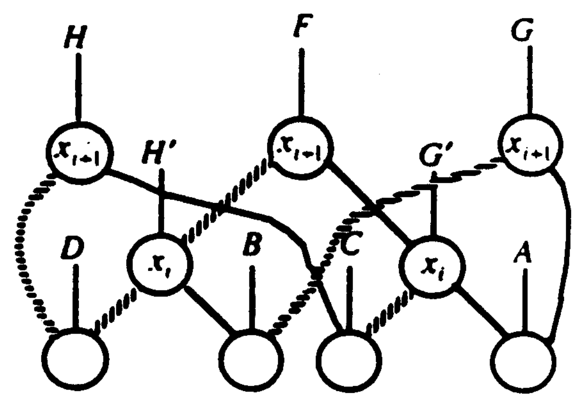
For all levels above x_i,
the set of cofactors remains unchanged by variable exchange because levels x_i and x_{i+1} are not involved

For all levels below x_{i+1},
the set of cofactors remains unchanged by variable exchange because of commutivity of cofactor

$$\left(F_{x_i}\right)_{x_{i+1}} = \left(F_{x_{i+1}}\right)_{x_i} = F_{x_i x_{i+1}}$$

Adjacent Variable Swap





Before variable swap:

$$F = (x_i, G, H) = (x_i, (x_{i+1}, A, B), (x_{i+1}, C, D))$$

After variable swap:

$$F = (x_{i+1}, G', H') = (x_{i+1}, (x_i, A, C), (x_i, B, D))$$

Adjacent Variable Swap Comments

- Several special cases:
  G does not depend on x_{i+1} implies $A = B$
  H does not depend on x_{i+1} implies $C = D$
  If $A = C$, then $G' = (x_i, A, C) = A$
  If $B = D$, then $H' = (x_i, B, D) = B$

- Modification of F for the new variable order:
  - removes, at most, nodes G and H from the DAG
    these nodes may be deleted if they are referenced only by F
  - adds, at most, nodes G' and H' to the DAG
    these nodes may be redundant or may already exist in the DAG

Complexity of Adjacent Variable Swap

- Overwrite each node at level i with a new node at level $i+1$

  $$(x_i, G, H) \rightarrow (x_{i+1}, G', H')$$
  $$(x_i, (x_{i+1}, A, B), (x_{i+1}, C, D)) \rightarrow (x_{i+1}, (x_i, A, C), (x_i, B, D))$$

- How to reach nodes at level i?
  - walk DAG from the roots to reach level i (expensive in run-time)
  - double-linked list for all nodes at each level (expensive in memory)
  - replace unique table with an array of hash tables, one per level
    replace
    ```
    hash_lookup(unique_table, (i,G,H), &value)
    ```
    with
    ```
    hash_lookup(unique_table[i], (G,H), &value)
    ```
  walk down the hash table array for each level to reach all nodes

- Adjacent variable swap complexity is proportional to the number of nodes at level i and independent of the total DAG size!

Window Permutation Algorithm

- Exhaustive search of all orders within a limited size window

- e.g., variables $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$. window size 3 starting at x_3

Start	$(x_1, x_2, \mathbf{x_3}, \mathbf{x_4}, \mathbf{x_5}, x_6, x_7)$
swap (x_3, x_4)	$(x_1, x_2, \mathbf{x_4}, \mathbf{x_3}, \mathbf{x_5}, x_6, x_7)$
swap (x_3, x_5)	$(x_1, x_2, \mathbf{x_4}, \mathbf{x_5}, \mathbf{x_3}, x_6, x_7)$
swap (x_4, x_5)	$(x_1, x_2, \mathbf{x_5}, \mathbf{x_4}, \mathbf{x_3}, x_6, x_7)$
swap (x_4, x_3)	$(x_1, x_2, \mathbf{x_5}, \mathbf{x_3}, \mathbf{x_4}, x_6, x_7)$
swap (x_5, x_3)	$(x_1, x_2, \mathbf{x_3}, \mathbf{x_5}, \mathbf{x_4}, x_6, x_7)$

- Repeat optimal search within the window at each variable position

[Fujita et al. EDAC'91]
[Ishiura et al. ICCAD'91]

Window Permutation Algorithm

- Move window of size k to a spot in the DAG
  - Explore all $k!$ permutations using $k!-1$ adjacent variable swaps
  - Record best permutation seen along the way
  - Restore optimal permutation with at most $k(k-1)/2$ adjacent swaps

- Iterate sliding window across the variables while DAG size decreases
  - Local optimum condition: a variable has to move at least k positions to reduce the DAG size

- Key limitation: can only afford small windows (e.g., $k \leq 5$)

Sifting Algorithm

- Find the optimum position for a variable assuming other variables remain fixed
- e.g., 7 positions for x_4 (including its current position)

$$\downarrow x_1 \downarrow x_2 \downarrow x_3 \;\; \mathbf{x_4} \;\; x_5 \downarrow x_6 \downarrow x_7 \downarrow$$

- Use pairwise adjacent swap to exhaustively search all 7 positions

start	$(x_1, x_2, x_3, \mathbf{x_4}, x_5, x_6, x_7)$
swap x_4, x_5	$(x_1, x_2, x_3, x_5, \mathbf{x_4}, x_6, x_7)$
swap x_4, x_6	$(x_1, x_2, x_3, x_5, x_6, \mathbf{x_4}, x_7)$
swap x_4, x_7	$(x_1, x_2, x_3, x_5, x_6, x_7, \mathbf{x_4})$
swap x_7, x_4	$(x_1, x_2, x_3, x_5, x_6, \mathbf{x_4}, x_7)$
swap x_6, x_4	$(x_1, x_2, x_3, x_5, \mathbf{x_4}, x_6, x_7)$
swap x_5, x_4	$(x_1, x_2, x_3, \mathbf{x_4}, x_5, x_6, x_7)$
swap x_3, x_4	$(x_1, x_2, \mathbf{x_4}, x_3, x_5, x_6, x_7)$
swap x_2, x_4	$(x_1, \mathbf{x_4}, x_2, x_3, x_5, x_6, x_7)$
swap x_1, x_4	$(\mathbf{x_4}, x_1, x_2, x_3, x_5, x_6, x_7)$

Sifting Algorithm Comments

- Sift each variable from its current position
  - down to the bottom of the DAG, then up to the top
  - record best position seen
  - restore best position after completing search
- Advantages:
  - variables can move an arbitrarily long distance independent of intermediate increases in the DAG size
  - solves Achilles heel ordering problem optimally starting with bad order
  - solves adder ordering problem optimally starting with bad order

Dynamic Variable Ordering – Results

- Experiment #1: Window Permutation Algorithm vs. Sift Algorithm
  - 35 largest examples from IWLS'91 benchmark set
  - 100,000 node limit placed on the OBDD package
    if memory limit exceeded, example is *unsolved*
  - Attempt to form OBDD for all primary outputs
    start with heuristic depth-first variable order
      (11 of 35 circuits are unsolved without dynamic ordering)
    apply BDD minimization every time DAG doubles in size

- Results:
  window permutation algorithm:
    k=2: solves 2 of 11 unsolved problems
    k=3: solves 3 of 11 unsolved problems
    k=4: solves 3 of 11 unsolved problems
  sift algorithm:
    solves 9 of 11 unsolved problems

Dynamic Variable Ordering – Results

- Experiment #2: Heuristic order vs. random order starting point
  - 35 largest examples from IWLS'91 benchmark set
  - 100,000 node limit placed on the OBDD package
    if memory limit exceeded, example is *unsolved*
  - Attempt to form OBDD for all primary outputs
    start with random variable order
      (23 of 35 examples are unsolved without dynamic ordering)
    apply BDD minimization every time DAG doubles in size

- Compare heuristic ordering start from random order start
  sift algorithm:
    solves 32 of 35 examples
    random fails for 1 example which succeeds with heuristic order
    2x longer run-time starting from random order
    slightly larger DAG sizes when starting from random order

Dynamic Variable Ordering Summary

- Effective technique to increase utility and application of OBDDs
  - allows OBDD computation to complete in many cases
  - classic space vs. time trade-off
    - no memory increase
    - run-time increases up to 10x

- (Almost) removes need for heuristic ordering algorithms

- Sifting algorithm superior to window permutation
  - produces smaller DAG sizes
  - allows more examples to complete

- Dynamic variable ordering future work
  - need to explore other applications to demonstrate utility
  - need faster and more effective BDD minimization algorithms

